Only the exercises with an asterisk (*) will be corrected.

6.1. MC questions.

(a) Which of the following statements are true?

☐ Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and p is a integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be function. If $g \circ f$ is continuous, then g is continuous or f is continuous.

Solution. False. We give a counterexample in one dimension, i.e. $m = n = p = 1$. With the same idea, one can build counterexamples for any dimension. Let $X = [0, 2]$, $Y = [0, 3]$. $f(x) = x$ for $x \in [0, 1]$ and $f(x) = x + 1$ for $x \in (1, 2]$. $g(x) = x$ for $x \in [0, 1]$ and $g(x) = x - 1$ for $x \in (1, 3]$. Clearly, $(g \circ f)(x) = x$ is continuous but neither g nor f is continuous.

☐ Let $X \subset \mathbb{R}^n$ be closed and p is a integer. If $f : X \to \mathbb{R}^p$ is a continuous function, then $f(X)$ is closed.

Solution. False. Note that $\mathbb{R}^n$ is closed. We define $f(x) = \frac{1}{\|x\|^2+1}$, but $f(\mathbb{R}^n) = (0, 1]$ is not closed.

(b) Which of the following sets are compact?

☐ $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 2022\}$;

Solution. A is bounded but not closed, because the boundary in not included.

☑ $B = \{(a, b, c) \in \mathbb{R}^3 \mid a, b, c \text{ are integers and } a^2 + b^2 + c^2 < 2022\}$;

Solution. B is compact because it contains finitely many points.

☐ $C = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in (0, 1], f(x) = \sin(\frac{1}{x})\}$;

Solution. C is not closed. $(\frac{1}{2k\pi}, 0) \in C$ for any positive integer k and $(\frac{1}{2k\pi}, 0) \to (0, 0)$ for $k \to +\infty$, but $(0, 0) \notin C$.

☐ $D = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid \theta \text{ is a rational number}\}$;

Solution. D is not closed. Clearly $(0, 1) \notin D$. Indeed, we know $(0, 1) = (\cos\theta, \sin\theta)$ if and only if $\theta = (2k + \frac{1}{2})\pi, k \in \mathbb{Z}$, but $(2k + \frac{1}{2})\pi$ is never a rational number for any $k \in \mathbb{Z}$. Since any irrational can be the limit of a rational sequence, i.e. there exists a rational sequence $\theta_k, k \in \mathbb{N}$ with $\lim_{k\to+\infty} \theta_k = \frac{\pi}{2}$, by continuity we have $\lim_{k\to+\infty}(\cos\theta_k, \sin\theta_k) = (1, 0)$. Hence D is not closed.

☑ $E = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \le 2\}$.

Solution. E is closed and bounded, hence compact.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. For f to be continuous at $(0, 0)$, which of the following needs to hold?

Hint: it might be helpful to do Exercise 6.3 first.

☐ it is sufficient that, along *some* direction $v \neq 0$, the directional derivative $D_v f(0, 0)$ exists.

☐ it is sufficient that *both* the partial derivatives $\partial_x f(0, 0)$ and $\partial_y f(0, 0)$ exist.

☐ it is sufficient that the directional derivatives $D_v f(0, 0)$ along *every* direction $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ exists.

Solution. All the statements are false. We consider:

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

The function f is not continuous at 0. However, $D_v f(0, 0)$ exists along every direction; see Exercise 6.3 below.

***6.2. Jacobi matrix** Compute the Jacobi matrix of the following functions:

(a)

$$f : \mathbb{R}^3 \to \mathbb{R}^2\,, \quad (u, v, w) \mapsto \begin{pmatrix} uv \\ w \end{pmatrix}.$$

(b)

$$g : \mathbb{R}^2 \to \mathbb{R}^3\,, \quad (x, y) \mapsto \begin{pmatrix} x^2 + e^y \\ x + y \\ y \end{pmatrix}$$

(c)

$$h : \mathbb{R}^3 \to \mathbb{R}^3; \qquad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\,\cos(\theta)\cos(\phi) \\ r\,\cos(\theta)\sin(\phi) \\ r\,\sin(\theta) \end{pmatrix}.$$

Solution.

(a)

$$J_f\begin{pmatrix}u\\v\\w\end{pmatrix} = \begin{pmatrix} v & u & 0\\ 0 & 0 & 1\end{pmatrix}.$$

(b)

$$J_g\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 2x & e^y\\ 1 & 1\\ 0 & 1\end{pmatrix}$$

(c)

$$J_f\begin{pmatrix}r\\\theta\\\phi\end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\phi) & -r\sin(\theta)\cos(\phi) & -r\cos(\theta)\sin(\phi)\\ \cos(\theta)\sin(\phi) & -r\sin(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi)\\ \sin(\theta) & r\cos(\theta) & 0\end{pmatrix}.$$

*6.3. Partial derivatives vs. differentiability.

Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a function and $v \in \mathbb{R}^n$ a vector. When it exists, the limit

$$D_v g(x) = \lim_{h\to 0} \frac{g(x+hv)-g(x)}{h}$$

is called the *directional derivative* of g along v at the point $x \in \mathbb{R}^n$. In particular, along the coordinate directions $e_i = (0,\dots,0,1,0,\dots,0)$ we have $D_{e_i} g = \frac{\partial g}{\partial x_i} = \partial_{x_i} g$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ to be the following function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0)\\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that, for any point $a \in \mathbb{R}^2$ and any direction $u \in \mathbb{R}^2$, f admits a directional derivative $D_u f(a)$.

Hint: To prove $D_u f(a)$ exists, one needs to show that $t \mapsto f(a+tu)$ is differentiable at the point $t = 0$. Recall the definition of differentiability in one variable and use the value $f(0,0)$.

(b) Show that f is not differentiable at the point $(0,0)$.

Hint: Recall that, when f is differentiable at the point $a \in \mathbb{R}^2$, f is also continuous at a.

Solution.

(a) Let $a = (x_0, y_0) \neq (0,0)$ and $u = (u_1, u_2) \neq (0,0)$, then we have

$$f(a+tu) = \frac{(x_0+tu_1)^2(y_0+tu_2)}{(x_0+tu_1)^4 + (y_0+tu_2)^2}.$$

The denominator is non-zero and bounded away from 0 from below. Then by composition rule, we have $t \mapsto f(a+tu)$ is differentiable at the point $t = 0$. Now we have showed that f admits all directional derivatives at $a \neq (0,0)$.

Next we prove, all directional derivatives at the point $a = (0,0)$ exist. First note that f vanishes on the line $\{(x,y) \in \mathbb{R}^2 \mid y = 0\}$. It follows that for $u = (u_1, 0) \neq 0$

$$f((0,0)+tu) = f(tu) = f(tu_1, 0) = 0.$$

Hence $D_u f(0,0) = 0$ for $u = (u_1, 0)$.

On the other hand, for any $u = (u_1, u_2)$ with $u_2 \neq 0$, one can write the vector u using polar coordinates as $u = (r\cos(\theta), r\sin(\theta))$ with $\sin(\theta) \neq 0$. Then for such a direction given by vector u we have

$$f((0,0)+tu) = f(tu) = \frac{t^3r^3\cos^2(\theta)\sin(\theta)}{t^4r^4\cos^4(\theta) + t^2r^2\sin^2(\theta)} = \frac{tr\cos^2(\theta)\sin(\theta)}{t^2r^2\cos^4(\theta) + \sin^2(\theta)}.$$

Again since $\sin^2(\theta) \neq 0$, we have $D_u f(0,0) = \frac{d}{dt}\big|_{t=0} f(tu) = \frac{r\cos^2(\theta)\sin(\theta)}{\sin^2(\theta)} = \frac{r\cos^2(\theta)}{\sin(\theta)}$. Hence $D_u f(0,0)$ also exists for any $u = (u_1, u_2)$ with $u_2 \neq 0$ and this concludes our proof.

(b) Note that f vanishes on the line $\{(x,y) \mid y = 0,\ x \neq 0\}$. Then if the limit exists, we know $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. However, for $y = x^2$ we can see

$$f(x, x^2) = \frac{x^4}{x^4 + x^4} = \frac{1}{2}.$$

The curve $\{(x, x^2) \mid x \in \mathbb{R}\}$ converges to $(0,0)$ as $x \to 0$, thus f is not continuous at $(0,0)$.

6.4. Tangent plane. Given the function

$$\begin{aligned} f\colon \Omega &\to \mathbb{R} \\ (x,y) &\mapsto \sqrt{1-x^2-y^2}, \end{aligned}$$

with $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$, compute the tangent plane of the graph of f at the points $(0,0)$, $\left(\frac{\sqrt{2}}{2},0\right)$, and $\left(0,\frac{1}{2}\right)$.

Solution. We compute

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{-x}{\sqrt{1-x^2-y^2}} \\ \frac{\partial f}{\partial y} &= \frac{-y}{\sqrt{1-x^2-y^2}} \end{aligned}$$

and then the tangent plane at a point (x_0,y_0) is given by

$$\begin{aligned} z &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) \\ &= \sqrt{1-x_0^2-y_0^2} - \frac{x_0}{\sqrt{1-x_0^2-y_0^2}}(x-x_0) - \frac{y_0}{\sqrt{1-x_0^2-y_0^2}}(y-y_0). \end{aligned}$$

Substituting $(x_0,y_0)=(0,0)$, we get:

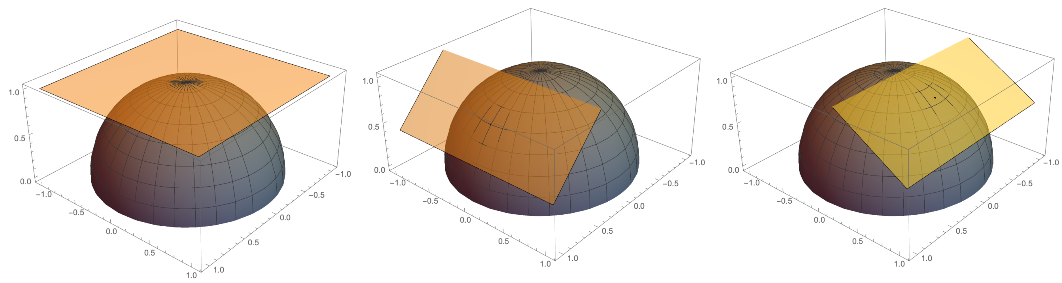
$$z = 1.$$

For $(x_0,y_0) = \left(\frac{\sqrt{2}}{2},0\right)$, we get

$$z = \sqrt{2} - x.$$

Finally, when $(x_0,y_0) = \left(0,\frac{1}{2}\right)$, we get:

$$z = \frac{1}{\sqrt{3}}(2-y).$$



(a) $(x_0, y_0) = (0, 0)$ (b) $(x_0, y_0) = \left(\frac{\sqrt{2}}{2}, 0\right)$ (c) $(x_0, y_0) = \left(0, \frac{1}{2}\right)$

Figure 1: Tangent planes to the graph of f at the point (x_0, y_0).