Only the exercises with an asterisk (*) will be corrected.

7.1. MC questions.

(a) Determine whether the following statement is true or false.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then, without doing any computation, we always know what is its differential $dL(x)$ at every point $x \in \mathbb{R}^m$.

☑ True ☐ False

Solution. By linearity we have $L(x+y) - L(x) - L(y) = 0$ for every $x, y \in \mathbb{R}^m$, and hence trivially

$$\lim_{y \to 0} \frac{|L(x+y) - L(x) - L(y)|}{|y|} = 0,$$

which means precisely that $dL(x) \equiv L$ for every $x \in \mathbb{R}^m$.

(b) Determine whether the following statements is true or false.

Let $M_{n\times n}(\mathbb{R})$ be the space of $n \times n$ matrices which we identify with the Euclidean space $\mathbb{R}^{n^2}$. The function "determinant" $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$, is

		True	False
(a)	is continuous on $M_{n\times n}(\mathbb{R})$	☑	☐
(b)	continuous only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐
(c)	differentiable on $M_{n\times n}(\mathbb{R})$	☑	☐
(d)	differentiable only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐

Solution. The only correct answers are: (a) is continuous on $M_{n\times n}(\mathbb{R})$, and (c) differentiable on $M_{n\times n}(\mathbb{R})$.

Indeed, if $A = (a_{ij})_{i,j}$ is a matrix, we may compute with respect to the first row

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(\widehat{A}_{1j}),$$

where $\det(\widehat{A}_{1j})$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the 1st row and the jth column. From this formula and by induction on n one sees that $\det A$ is just a polynomial in the variables a_{ij}, and so differentiable and *a fortiori* continuous.

***7.2. Calculating the differential.**

Calculate $df(x_0)[v]$ for

(a) $f(x) = x^2, x_0 = 3, v = 5.$

Solution. $df(x_0)[v] = 30.$

(b) $f(x) = y\cos^2(x), x_0 = (\pi/4, 1), v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solution. We have:

$$df(x) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = \left(-2\cos(x)\sin(x)y,\ \cos^2(x)\right)$$

and therefore

$$df(x_0)[v] = \cos^2(\pi/4) = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}.$$

(c) $f(x) = e^x\ln(y) + z, x_0 = (0,1,0), v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Solution. We have

$$df(x) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right) = \left(e^x\ln(y),\ e^x/y,\ 1\right),$$

and therefore

$$df(x_0)[v] = -e^0\ln(1) + 0\cdot e^0/1 + 1 = 1.$$

7.3. Gradient and Level Sets.

A *curve in the plane* is a subset $\Gamma \subset \mathbb{R}^2$ so that there is a differentiable function $\gamma : (a,b) \subset \mathbb{R} \to \mathbb{R}^2$ so that $\gamma((a,b)) = \Gamma$ and $\gamma'(t) \neq 0$ for every $t \in (a,b)$. Any such γ is called *parametrization* of Γ.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function so that its *level set* at c

$$\Gamma = f^{-1}(\{c\}) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$$

is a curve in the plane, and let $\gamma : I = (a,b) \to \Gamma$ be a parametrization. Prove that:

(a) The gradient to f is orthogonal to Γ, namely

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \text{for every } t \in I.$$

(b) The directional derivative of f along Γ vanishes, namely

$$D_{\gamma'(t)} f(\gamma(t)) = 0 \quad \text{for every } t \in I.$$

(c) At a point $(x, y) \in \Gamma$, the direction where f grows the most is orthogonal to Γ (this means: among all the vectors v with $|v| = 1$, $D_v f(x, y)$ assumes the maximum value when v is orthogonal to Γ).

Solution.

(a) Since $I \ni t \mapsto f \circ \gamma(t)$ is constant, the chain rule yields that for every $t \in I$ there holds

$$0 = \frac{d}{dt} f \circ \gamma(t) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t).$$

(b) By definition of gradient, we have for every $t \in I$

$$D_{\gamma'(t)} f(\gamma(t)) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = 0.$$

(c) We need to find the unit vector v so that $D_v f(x, y)$ is largest. Let $t \in I$ be so that $\gamma(t) = (x, y)$. Since $\gamma'(t) \neq 0$, we may choose a unit vector normal to $\gamma'(t)$, namely $\mathsf{n} \in \mathbb{R}^2$ with $\|\mathsf{n}\| = 1$ and $\mathsf{n} \cdot \gamma'(t) = 0$, any other vector $v \in \mathbb{R}^2$ may be written uniquely as

$$v = a\gamma'(t) + b\mathsf{n},$$

for some $a, b \in \mathbb{R}$.

From (b) we have that $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$ and so

$$\begin{aligned} D_v f(\gamma(t)) &= \nabla f(\gamma(t)) \cdot v \\ &= \nabla f(\gamma(t)) \cdot (a\gamma'(t) + b\mathsf{n}) \\ &= b\Big(\nabla f(\gamma(t)) \cdot \mathsf{n}\Big). \end{aligned}$$

Hence, since $|v| = 1$, the maximum of $D_v f(x, y)$ is achieved when either $b = 1$ if $\nabla f(\gamma(t)) \cdot \mathsf{n} \geq 0$ or $b = -1$ if $\nabla f(\gamma(t)) \cdot \mathsf{n} < 0$. In any case it must necessarily be $a = 0$. This means that v is orthogonal to Γ.

***7.4. Calculations in polar coordinates.**

The polar coordinates on $\mathbb{R}^2$ are defined via the map

$$P : (0,\infty) \times \mathbb{R} \to \mathbb{R}^2, \qquad P(r,\phi) := \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}.$$

(a) Show that P is continuously differentiable and calculate the derivative dP.

(b) Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be a differentiable function and define $g(r,\phi) := f(P(r,\phi))$. Show that

$$\partial_x f(P(r,\phi)) = \partial_r g(r,\phi)\cos(\phi) - \frac{1}{r}\sin(\phi)\partial_\phi g(r,\phi)$$
$$\partial_y f(P(r,\phi)) = \partial_r g(r,\phi)\sin(\phi) + \frac{1}{r}\cos(\phi)\partial_\phi g(r,\phi).$$

holds and conclude that

$$\nabla f(P(r,\phi)) = \partial_r g(r,\phi) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} + \frac{1}{r}\partial_\phi g(r,\phi) \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix}.$$

Hint: *Use the chain rule.*

(c) Let

$$f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}, \qquad f(x,y) := \log(\sqrt{x^2+y^2}).$$

Use the chain rule and part (b) to verify that

$$\nabla f(x,y) = \frac{1}{x^2+y^2} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hint: *the function f is in polar coordinates given by $g(r,\phi) = \log(r)$.*

Solution.

(a) The derivative of P is given by the Jacobi matrix

$$dP(r,\phi) = \begin{pmatrix} \partial_r P_1 & \partial_\phi P_1 \\ \partial_r P_2 & \partial_\phi P_2 \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{pmatrix}.$$

Since all the partial derivatives are continuous, the map P is continuously differentiable.

(b) The chain rule gives $df \circ dP = dg$, that is,

$$(\partial_x f(P(r,\phi)), \partial_y f(P(r,\phi))) \begin{pmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{pmatrix} = (\partial_r g(r,\phi), \partial_\phi g(r,\phi)).$$

We use

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

to invert $dP(r,\phi)$ and obtain:

$$(\partial_x f(P(r,\phi)), \partial_y f(P(r,\phi))) = (\partial_r g(r,\phi), \partial_\phi g(r,\phi)) \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ \frac{-1}{r}\sin(\phi) & \frac{1}{r}\cos(\phi) \end{pmatrix}.$$

After multiplying the right-hand side, we obtain the wanted formulae:

$$\begin{aligned} \partial_x f(P(r,\phi)) &= \partial_r g(r,\phi)\cos(\phi) - \frac{1}{r}\sin(\phi)\partial_\phi g(r,\phi) \\ \partial_y f(P(r,\phi)) &= \partial_r g(r,\phi)\sin(\phi) + \frac{1}{r}\cos(\phi)\partial_\phi g(r,\phi). \end{aligned}$$

(c) Let $f(x,y) = \log(\sqrt{x^2+y^2})$ and $g(r,\phi) := f(P(r,\phi)) = \log(r)$. (Here we use $(r\cos(\phi))^2 + (r\sin(\phi))^2 = r^2(\cos^2(\phi)+\sin^2(\phi)) = r^2$). Since $\partial_\phi g = 0$ holds, using the formula from the previous part, we obtain:

$$\nabla f(x,y) = \partial_r g(r,\phi) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

where $(x,y) = P(r,\phi)$ holds. We can rewrite the last expression in Cartesian coordinates to obtain:

$$\nabla f(x,y) = \frac{1}{x^2+y^2} \begin{pmatrix} x \\ y \end{pmatrix}.$$