Only the exercises with an asterisk (*) will be corrected.

8.1. MC questions.

(a) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a C^k map with $k \geq 2$. Which of the following statements are true?

- ☐ The gradient $\nabla f(x)$ is an $n \times n$ matrix.

  Solution. The gradient $\nabla f(x)$ is an n-dimensional vector, so, unless $n = 1$, it is not an $n \times n$ matrix.

- ☑ The Hessian matrix $\mathrm{Hess}_f(x)$ is a square matrix.

- ☑ $\mathrm{Hess}_f(x)$ is symmetric.

  Solution. See Proposition 3.5.4 and Definition 3.5.8 in the script.

- ☐ $\mathrm{Hess}_f(x)$ is invertible.

  Solution. A counterexample can be found in Question 7.2 below.

(b) Consider the function $f(x, y) := (x^2 + y^2)e^{xy}$. Which statements are correct?

- ☑ The Hessian matrix of f at $(0, 0)$ is positive definite.

  Solution. The Hessian matrix of f at $(x, y) = (0, 0)$ is given by

$$\mathrm{Hess}_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

  For any vector $v \in \mathbb{R}^2 \setminus (0, 0)$, we have $v^T \mathrm{Hess}_f(0, 0)v = \|v\| > 0$.

- ☐ The Hessian matrix of f at $(0, 0)$ is negative definite.
- ☐ The Hessian matrix of f at $(0, 0)$ has positive and negative eigenvalues.

8.2. Hessian matrix.

Calculate the Hessian matrix of the following function $f: \mathbb{R}^2 \to \mathbb{R}$ at the point (x_0, y_0):

(a) $f(x, y) = x^2 + xy + y^2, \quad (x_0, y_0) = (1, 1),$

(b) $f(x, y) = \cos(x) + \sin(y)x, \quad (x_0, y_0) = (\pi/2, \pi/2),$

(c) $f(x, y) = y^3 \cdot \left(\sin\left(e^{y^5 \cdot \sin(y^2)}\right) \cdot \cos(\sinh(y))\right), \quad (x_0, y_0) = (0, 0).$

Solution.

(a) We calculate all partial derivatives of the second order and we note that, a priori, $\partial_{xy} f = \partial_{yx} f$ holds because the given function f is C^∞ (in particular f is C^2).

$$\begin{aligned}
&\partial_{xx} f(x,y) = \partial_x (2x+y) = 2,\\
&\partial_{yy} f(x,y) = 2,\\
&\partial_{xy} f(x,y) = \partial_x (x+y) = 1 = \partial_{yx} f(x,y).
\end{aligned}$$

With this, we obtain that the Hessian matrix is constant, in particular: $\text{Hesse}_f(x,y) = \text{Hesse}_f(1,1)$ with

$$\text{Hesse}_f(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(b) Similarly as above, we calculate:

$$\begin{aligned}
&\partial_{xx} f(x,y) = -\cos(x),\\
&\partial_{yy} f(x,y) = -\sin(y)x,\\
&\partial_{xy} f(x,y) = \cos(y) = \partial_{yx} f(x,y),
\end{aligned}$$

and thus

$$\text{Hesse}_f(x,y) = \begin{pmatrix} -\cos(x) & \cos(y) \\ \cos(y) & -\sin(y)x \end{pmatrix}.$$

In particular:

$$\text{Hesse}_f(\pi/2,\pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & -\pi/2 \end{pmatrix}.$$

(c) The function f is of the form

$$f(x, y) = y^3 \cdot g(y).$$

In other words, $f(x, y)$ is independent of x, which implies

$$\partial_{xy} f(x, y) = \partial_{yx} f(x, y) = \partial_{xx} f(x, y) = 0.$$

Because we are only interested in the Hessian matrix at the point $(0, 0)$, we only have to calculate $\partial_{yy} f(0, 0)$. We have:

$$\begin{aligned} \partial_{yy} f(x, y) &= \partial_y (3y^2 \cdot g(y) + y^3 \cdot g'(y)) \\ &= 6y \cdot g(y) + 3y^2 g'(y) + 3y^2 \cdot g'(y) + y^3 \cdot g''(y), \end{aligned}$$

and independently of what $g(0), g'(0)$ and $g''(0)$ are, we obtain

$$\partial_{yy} f(x, y) = 0.$$

The Hessian matrix $\text{Hesse}_f(0, 0)$ is therefore the 2×2 zero-matrix.

*8.3. Taylor polynomials.

Consider the following function:

$$f(x, y) := e^{x^2+y^2} + \log(1 + x^2) + \arctan(xy).$$

(a) Determine the Taylor polynomial of f at $(0, 0)$ up to and including third order.

(b) Let

$$g\colon \mathbb{R} \to \mathbb{R}, \qquad g(x) := f(x, x).$$

Determine the Taylor polynomial of g at $(0, 0)$ up to and including third order.

What is the relationship between the Taylor polynomials of f and of g?

Solution.

(a) We have:

$$f(0, 0) = 1$$

The first partial derivatives are:

$$\partial_x f(x,y) = 2xe^{x^2+y^2} + \frac{2x}{1+x^2} + \frac{y}{1+x^2y^2}$$
$$\partial_y f(x,y) = 2ye^{x^2+y^2} + \frac{x}{1+x^2y^2}$$

Direct substitution gives us:

$$df(0,0) = (0,0)$$

We now calculate the partial derivatives of second order:

$$\partial_{xx} f(x,y) = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} + \frac{2(1+x^2)-4x^2}{(1+x^2)^2} - \frac{2xy^3}{(1+x^2y^2)^2}$$
$$\partial_{xy} f(x,y) = 4xye^{x^2+y^2} + \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2}$$
$$\partial_{yy} f(x,y) = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2} - \frac{2x^3y}{(1+x^2y^2)^2}$$

We know $\partial_{xy} f(x,y) = \partial_{yx} f(x,y)$. Substituting, we get:

$$Hess_f(0,0) = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

We now have to calculate the partial derivatives of third order. These all vanish at $(0,0)$. We write out the calculation below in four cases; the other ones follow by symmetry:

$$\begin{aligned}\partial_{xxx} f(x,y) = {} & 12xe^{x^2+y^2} + 8x^3e^{x^2+y^2} + 2\frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4} \\ & - \frac{2y^3(1+x^2y^2)^2 - 8x^2y^5(1+x^2y^2)}{(1+x^2y^2)^4}\end{aligned}$$
$$\partial_{xxy} f(x,y) = 4ye^{x^2+y^2} + 8x^2ye^{x^2+y^2} - \frac{6xy^2(1+x^2y^2)^2 - 8x^3y^4(1+x^2y^2)}{(1+x^2y^2)^4}$$
$$\partial_{xyy} f(x,y) = 4xe^{x^2+y^2} + 8xy^2e^{x^2+y^2} - \frac{6x^2y(1+x^2y^2)^2 - 8x^4y^3(1+x^2y^2)}{(1+x^2y^2)^4}$$
$$\partial_{yyy} f(x,y) = 12ye^{x^2+y^2} + 8y^3e^{x^2+y^2} - \frac{2x^3(1+x^2y^2)^2 - 8x^5y^2(1+x^2y^2)}{(1+x^2y^2)^4}$$

At $(0,0)$, all third order partial derivatives vanish. It therefore follows:

$$\begin{aligned} T_3f((0,0);(x,y)) &= f(0,0) + \partial_x f(0,0)x + \partial_y f(0,0)y + \frac{1}{2}\partial_{xx}f(0,0)x^2 \\ &+ \partial_{xy}f(0,0)xy + \frac{1}{2}\partial_{yy}f(0,0)y^2 + \frac{1}{6}\partial_{xxx}f(0,0)x^3 + \frac{1}{2}\partial_{xxy}f(0,0)x^2y \\ &+ \frac{1}{2}\partial_{xyy}f(0,0)xy^2 + \frac{1}{6}\partial_{yyy}f(0,0)y^3 \\ &= 1 + 2x^2 + xy + y^2 \end{aligned}$$

(b) We know the Taylor polynomials of the following functions:

$$\begin{aligned} e^y &\simeq 1 + y + \frac{y^2}{2} \\ \log(1+y) &\simeq y - \frac{y^2}{2} \\ \arctan(y) &\simeq y \end{aligned}$$

After a change of variables, we obtain:

$$\begin{aligned} e^{2x^2} &\simeq 1 + 2x^2 + \frac{4x^4}{2} \\ \log(1+x^2) &\simeq x^2 - \frac{x^4}{2} \\ \arctan(x^2) &\simeq x^2 \end{aligned}$$

These are Taylor polynomials of 4th degree. We have

$$\begin{aligned} g(x) &= f(x,x) \\ &= e^{2x^2} + \log(1+x^2) + \arctan(x^2) \\ &\simeq 1 + 4x^2 = T_g^3(x;0), \end{aligned}$$

which gives us the Taylor polynomial of g of third order.

We could have alternatively also calculated the Taylor polynomial directly: in the Taylor polynomial of f of third degree, we substitute $x = y$ to obtain:

$$T_3f((0,0),(x,x)) = 1 + 4x^2.$$

The relationship between the Taylor polynomials of f and g is therefore given by:

$$T_3f((0,0),(x,x)) = T_g^3(x;0).$$

***8.4. Taylor approximation.**

Approximate the function

$$f(x, y) = e^x \sin y$$

at the point $(0, \pi/2)$ by a Taylor polynomial of first order and of second order. Use these Taylor polynomials to give an approximation of $f\left(0,\ \frac{\pi}{2} + \frac{1}{4}\right)$.

Solution.

The Taylor polynomial of first order at (x_0, y_0) is given by

$$T_1 f((x_0, y_0); (x - x_0, y - y_0)) = f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$$

and the Taylor polynomial of second order by

$$\begin{aligned} T_2 f((x_0, y_0); (x - x_0, y - y_0)) = &f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0) \\ &+ \frac{1}{2} f_{xx}(x_0, y_0) \cdot (x - x_0)^2 + \frac{1}{2} f_{yy}(x_0, y_0) \cdot (y - y_0)^2 \\ &+ f_{xy}(x_0, y_0) \cdot (x - x_0)(y - y_0) \end{aligned}$$

For the partial derivatives, we get:

$$\begin{aligned} &f_x(x, y) = f_{xx}(x, y) = e^x \sin y, \quad f_y(x, y) = e^x \cos y, \\ &f_{xy}(x, y) = e^x \cos y, \quad f_{yy}(x, y) = -e^x \sin y. \end{aligned}$$

and so at the point $(0, \pi/2)$:

$$\begin{aligned} &f(0, \pi/2) = f_x(0, \pi/2) = f_{xx}(0, \pi/2) = 1, \quad f_y(0, \pi/2) = f_{xy}(0, \pi/2) = 0, \\ &f_{yy}(0, \pi/2) = -1. \end{aligned}$$

The Taylor polynomial of the first order is therefore

$$T_1 f((0, \pi/2); (x - 0, y - \pi/2)) = 1 + 1 \cdot (x - 0) + 0 \cdot (y - \pi/2) = 1 + x$$

and the Taylor polynomial of the second order is:

$$T_2 f((0, \pi/2); (x - 0, y - \pi/2)) = 1 + x + \frac{1}{2} x^2 - \frac{1}{2} (y - \pi/2)^2.$$

For the point $(0, \pi/2 + \frac{1}{4})$ we get the approximation

$$T_1 f\left((0, \pi/2); \left(0, \frac{1}{4}\right)\right) = 1, \quad T_2 f\left((0, \pi/2); \left(0, \frac{1}{4}\right)\right) = 1 - \frac{1}{2} \cdot \left(\frac{1}{4}\right)^2 = \frac{31}{32} = 0.96875$$

The exact value of f (up to 6 decimals) is

$$f(0, \pi/2 + 1/4) = e^0 \sin(\pi/2 + 1/4) = 0.968912.$$

We see that our approximation differs from the actual value by less than 0.001.