Only the exercises with an asterisk (*) will be corrected.

9.1. MC questions.

(a) Choose the correct statement. Motivate your answer.

Recall that a *critical point* of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an $x_0 \in \mathbb{R}^n$ so that $df(x_0) = 0$. At such point, the tangent plane to the graph of f is:

- ☐ not defined
- ☑ horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis)
- ☐ vertical (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis)
- ☐ none of the above, in general.

Solution. The equation of the plane is

$$z = f(x_0) + df(x_0) \cdot (x, y) = f(x_0)$$

which means that it is parallel to the x-y plane, and therefore horizontal.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function and consider its restriction on the square $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

- ☐ If f has a local maximum, minimum, or a saddle point at x_0 in Q, then $df(x_0) = 0$.
- ☑ Let $x_0 \in Q$ be a point such that $df(x_0) = 0$, then f has a local max/min/saddle at x_0.

Solution. Consider for instance $f(x, y) = x$: clearly $f \leq 1$ on Q and so f has a maximum at each point in $(1, y)$; however $df(1, y) = (1, 0) \neq 0$, so the first statement is false.

However, if $df(x_0) = 0$, this means that f has a minimum, a maximum, or a saddle point at x_0 in $\mathbb{R}^2$, and thus so it is for the restriction of f to Q.

*9.2. Critical points

Find the critical points of the following functions and determine whether they are local minima, local maxima, or saddle points.

(a) $f\colon \mathbb{R}^2 \cap \{(x, y) \mid x > 0,\ y > 0\} \to \mathbb{R}, \qquad f(x, y) = \dfrac{y}{2x} + \dfrac{x - 1}{y^2}$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = x^3 + y^3 + 3xy$

(c) $f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad f(x,y,z) = x^2 + y^2 + z^2 + 2xyz.$

Solution.

(a) We have

$$df(x,y) = \left(-\frac{y}{2x^2} + \frac{1}{y^2}, \frac{1}{2x} - \frac{2(x-1)}{y^3}\right) = (0,0)$$

if and only if

$$\begin{cases} -\frac{1}{2}y^3 + x^2 = 0 \\ \frac{1}{2}y^3 - 2x(x-1) = 0 \end{cases}$$

which yields (as $x > 0$ and y is real)

$$(x,y) = (2,2).$$

Now, we have

$$d^2 f(x,y) = \begin{pmatrix} \frac{y}{x^3} & -\frac{1}{2x^2} - \frac{2}{y^3} \\ -\frac{1}{2x^2} - \frac{2}{y^3} & \frac{6(x-1)}{y^4} \end{pmatrix}.$$

Therefore, we have

$$d^2 f(2,2) = \begin{pmatrix} \frac{1}{4} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

and

$$\det d^2 f(2,2) = -\frac{3}{64} < 0$$

which implies that $d^2 f(2,2)$ admits a strictly positive eigenvalue and a strictly negative eigenvalue. We conclude that $(2,2)$ is a saddle point of f.

(b) We know

$$\nabla f(x,y) = (3x^2 + 3y, 3y^2 + 3x) = 0$$

if and only if $x^2 = -y$ and $y^2 = -x$. Then $x^4 = y^2 = -x$, also $x(x^3+1) = 0$ and $y(y^3+1) = 0$. The real solution of the equation $X(X^3+1)$ are $X = 0, -1$, so the solution of the system $x^2 = -y$, $y^2 = -x$ is given by

$$(0,0), \quad \text{and} \quad (x,y) = (-1,-1).$$

We have

$$\operatorname{Hess} f(x,y) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$$

also

$$\operatorname{Hess} f(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \operatorname{Hess} f(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$$

Note that $(0,0)$ is neither local minimum nor local maximum, since $\operatorname{Hess} f(0,0)$ has a positive eigenvalue 3 and a negative eigenvalue -3. $(-1,-1)$ is a local maximum, since $\operatorname{Hess} f(-1,1)$ has two negative eigenvalues -3 and -9.

(c) Compute the gradient

$$\nabla f(x,y,z) = (2x+2yz, 2y+2xz, 2z+2xy).$$

and $\nabla f(x,y,z) = 0$ implies

$$2x+2yz=0, \qquad 2y+2xz=0, \qquad 2z+2xy=0.$$

Then all the critical points are given by

$$\operatorname{Crit}(f) = \{(0,0,0), (-1,1,1), (1,-1,1), (1,1,-1), (-1,-1,-1)\}.$$

The Hessian matrix of f is given by

$$\operatorname{Hess}(f,(x,y,z)) = \begin{pmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{pmatrix}.$$

Hence the Hessian matrix at the point 0 is given

$$\operatorname{Hess}(f,(0,0,0)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is a postive Diagonal matrix and positive-definite, so $(0,0,0)$ is a local minimum.

The matrices

$$\operatorname{Hess}(f,(-1,1,1)) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}, \quad \operatorname{Hess}(f,(1,-1,1)) = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

$$\operatorname{Hess}(f,(1,1,-1)) = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad \operatorname{Hess}(f,(-1,-1,-1)) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

have the same determinant -32. We can easily see they are indefinite. Indeed, negative determinant means that their three eigenvalues are nonzero and the product of the eigenvalues is negative. However, the three eigenvalues cannot be all negative since the their traces are positive, so the only possibility is that they have 2 positive eigenvalues and 1 negative eigenvalue. Therefore, $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$ are saddle points.

9.3. Centre of mass.

Consider a system of N particles in $\mathbb{R}^n$, that is N points $a_1,\ldots,a_N$ with masses $m_1,\ldots,m_N$. Prove that the expression

$$I(x) = \sum_{i=1}^{N} m_i |x-a_i|^2$$

has a unique global minimum, and find it explicitly. Such point C is called the *centre of mass* of the system.

Solution. The gradient of I is

$$\nabla I(x) = \sum_{i=1}^{N} 2m_i \begin{pmatrix} x_1-(a_i)_1 \\ \cdots \\ x_n-(a_i)_n \end{pmatrix} = \sum_{i=1}^{N} 2m_i(x-a_i),$$

which vanishes exactly when

$$\begin{cases} \sum_i m_i x_1 = \sum_i m_i (a_i)_1, \\ \qquad \cdots \\ \sum_i m_i x_n = \sum_i m_i (a_i)_n \end{cases}$$

and hence, such critical point is

$$C = \frac{\sum_{i=1}^{N} m_i a_i}{\sum_{i=1}^{N} m_i}.$$

Now, I is *always positive*, *continuous* and $\lim_{x\to\infty} I(x) = +\infty$, which means that its global minimum has to be attained at some point in $\mathbb{R}^n$, which will then be critical. But the only critical point is C, so this has to be the global minimum.

One may also see this by computing the Hessian of I:

$$\text{Hess}_I(x) \equiv \sum_i 2m_i \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \Big(\sum_i 2m_i\Big)\mathbb{I}_{n\times n},$$

which is then positive definite everywhere. So C has to be a local minimum, and hence also the global minimum since $\lim_{x\to\infty} I(x) = +\infty$.

*9.4. Optimisation problem with constraints.

Determine the global extrema of the function

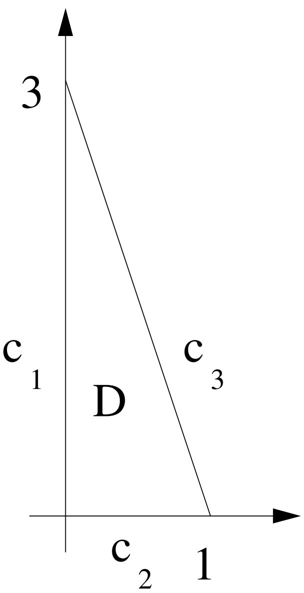
$$f(x,y) = x^2 + y^2 + 7x - 2y$$

on the set $D = \{(x,y) \mid x \geq 0,\ y \geq 0,\ 3x + y \leq 3\}$.

Hint: Try drawing D and consider the interior of D and the boundary separately.

Solution.

The set D consists of three corners $P_1 = (0,3),\ P_2 = (0,0),\ P_3 = (1,0)$, the lines connecting them c_1, c_2, c_3, and the interior $\widetilde{D}$.



The critical points of f in the interior $\widetilde{D}$ must satisfy the equation

$$Df(x,y) = (2x+7, 2y-2) \stackrel{!}{=} (0,0).$$

The point satisfying this equation is $(-\frac{7}{2}, 1)$ and it does not lie in the interior $\widetilde{D}$. Therefore the extrema of $f|_D$ lie on the boundary of D.

Along c_1, we have $x = 0$ and the possible critical points of f on c_1 are the critical points of the function

$$y \mapsto f(0,y) = y^2 - 2y.$$

From the equation $0 \stackrel{!}{=} \frac{d}{dy}(f(0,y)) = 2y - 2$ it follows that $y = 1$, i.e. we get the point $P_4 = (0,1) \in D$ with $f(P_4) = -1$ as the candidate for an extremum of f.

Along c_2, we have $y = 0$. Taking the derivative and finding critical points of the function

$$x \mapsto f(x,0) = x^2 + 7x$$

gives us $x = -\frac{7}{2}$. But the point $(-\frac{7}{2}, 0)$ does not belong to D.

Along c_3, we have $y = 3 - 3x$. We therefore need to determine the critical points of the function

$$x \mapsto f(x, 3-3x) = x^2 + (3-3x)^2 + 7x - 2(3-3x) = 10x^2 - 5x + 3.$$

The equation $0 \stackrel{!}{=} \frac{d}{dx}\Big(f(x, 3-3x)\Big) = 20x - 5$ gives us $x = \frac{1}{4}$, i.e. we get the point $P_5 = \left(\frac{1}{4}, 3 - \frac{3}{4}\right) = \left(\frac{1}{4}, \frac{9}{4}\right) \in D$ with

$$f(P_5) = 10\left(\frac{1}{4}\right)^2 - 5\left(\frac{1}{4}\right) + 3 = \frac{19}{8}$$

as a candidate for an extremal point of f.

We therefore have the following candidates:

	P_1	P_2	P_3	P_4	P_5
(x,y)	$(0,3)$	$(0,0)$	$(1,0)$	$(0,1)$	$(\frac{1}{4}, \frac{9}{4})$
$f(x,y)$	3	0	8	-1	$\frac{19}{8}$

and we see:

$$\min f|_D(x,y) = f(0,1) = -1 \quad \text{and} \quad \max f|_D(x,y) = f(1,0) = 8.$$