Only the exercises with an asterisk (*) will be corrected.

10.1. MC questions.

(a) Let $X \subseteq \mathbb{R}^n$ be an open set, $f\colon X \to \mathbb{R}^n$ a C^1 function, $\gamma\colon [a,b] \to X$ a parametrised path. Which of the following statements are true?

☑ If $g\colon X \to \mathbb{R}$ is a potential of f, then for every constant $C \in \mathbb{R}$, $h := g + C$ is also a potential of f.

Solution. True.

☑ The vector field f is conservative if and only only if f has a potential g.

Solution. True: every gradient vector field ∇g is conservative and the converse holds by Theorem 4.1.10 from the script.

☐ If X is star-shaped, then f is conservative.

Solution. False: generally f does not satisfy $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$, which is independent of X.

☐ If for every $i, j \in \{1, \ldots, n\}$ the equation $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ holds, then f is conservative.

Solution. False: an explicit counterexample is given by

$$f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad (x,y) \mapsto \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}.$$

The symmetry is fulfilled, but the path integral of f along the closed path $\gamma(t) = (\cos(t), \sin(t)),\ t \in [0, 2\pi]$ is not-zero. Therefore f is not conservative.

☐ Let $A_1, \ldots, A_m \subseteq X$ be open with $\bigcup_{k=1}^m A_k = X$. If $f|_{A_k}$ is conservative for all $k = 1, \ldots, m$, then f is conservative.

Solution. False: let f be as in the counterexample above. Let $A_0, \ldots, A_3$ be open sets in $\mathbb{R}^2 \setminus \{0\}$, each covering one of the four quadrants. The sets A_i can be chosen to be star-shaped and thus $f|_{A_i}$ is conservative for all $i = 0, \ldots, 3$. However, as we saw above, f is non-conservative on the whole of $\mathbb{R}^2 \setminus \{0\}$.

(b) The work W done by a force F on a body moving along a path γ is given by the line integral

$$W = \int_\gamma F(s) \cdot d\gamma.$$

Consider a spring that exerts a horizontal force $F \colon \mathbb{R}^2 \to \mathbb{R}^2, F(x, y) = (-kx, 0)$. Is the following statement true or false?

The work W done by this spring on a body moving along the path $\gamma(t) = (x(t), y(t))$ is given by

$$W = -\frac{kx^2}{2}.$$

☑ True.

☐ False.

Solution. The statement is true. We have

$$W = \int_\gamma F(s) \cdot d\gamma = \int_0^t \langle (-kx(s), 0), (x(s), y(s))' \rangle \, ds = \int_0^t -kx(s)x'(s)dt$$
$$= -\frac{1}{2}kx^2.$$

*10.2. Line integrals.

Compute the following line integrals.

(a) $v(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$, from $(-1, 1)$ to $(1, 1)$ along the curve $y = x^2$.

(b) $v(x, y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, from $(0, 0)$ to $(2, 0)$ along the curve $y = 1 - |1 - x|$.

(c) $v(x, y, z) = \begin{pmatrix} x \\ y \\ xz - y \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} t^2 \\ 2t \\ 4t^3 \end{pmatrix}$, $t \in [0, 1]$.

(d) $v(x, y) = \begin{pmatrix} 2a - y \\ x \end{pmatrix}$, along the curve $\gamma(t) = \begin{pmatrix} a(t - \sin(t)) \\ a(1 - \cos(t)) \end{pmatrix}$, $t \in [0, 2\pi]$, with a constant $a \in \mathbb{R}$.

Solution.

(a) A parametrization of the curve is given by $\gamma : [-1,1] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \Rightarrow \gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix},$$

then we have

$$v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = t^2 - 2t^3 + 2t^5 - 4t^4.$$

Now we can compute

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_{-1}^{1} t^2 - 2t^3 + 2t^5 - 4t^4 \; dt = \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5} \right]_{t=-1}^{1} \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left(\frac{-1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{4}{5} \right) \\ &= \frac{1}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{5} = \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}, \end{aligned}$$

(b) A parametrization of the curve is given by $\gamma(t) = (t, \gamma_2(t))$, $t \in [0,2]$, with

$$\gamma_2(t) = \begin{cases} t, & t \in [0,1] \\ 2-t, & t \in [1,2]. \end{cases}$$

then we have for $t \in [0,1]$

$$v(\gamma(t)) = \begin{pmatrix} 2t^2 \\ 0 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^2$$

and for $t \in [1,2]$

$$v(\gamma(t)) = \begin{pmatrix} t^2 + (2-t)^2 \\ t^2 - (2-t)^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2(2-t)^2$$

Now we can compute

$$\begin{aligned} \int_\gamma v \; d\gamma &= \int_0^1 2t^2 \; dt + \int_1^2 2(2-t)^2 dt = \left[\frac{2t^3}{3} \right]_{t=0}^{1} + 2 \left[\frac{-(2-t)^3}{3} \right]_{t=1}^{2} \\ &= \frac{2}{3} + 2\left(\frac{1}{3}\right) = \frac{4}{3}, \end{aligned}$$

(c) We have

$$v(\gamma(t)) = \begin{pmatrix} t^2 \\ 2t \\ 4t^5 - 2t \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} 2t \\ 2 \\ 12t^2 \end{pmatrix}, \quad v(\gamma(t)) \cdot \gamma'(t) = 2t^3 + 4t + 12t^2(4t^5 - 2t),$$

and this leads to

$$\int_\gamma v\ d\gamma = \int_0^1 2t^3 + 4t + 12t^2(4t^5 - 2t)\ dt = \int_0^1 4t + 48t^7 - 22t^3\ dt$$
$$= \left[2t^2 + 6t^8 - \frac{22t^4}{4}\right]_{t=0}^1 = 2 + 6 - \frac{11}{2} = \frac{4+12-11}{2} = \frac{5}{2}.$$

(d) We have

$$v(\gamma(t)) = \begin{pmatrix} 2a - a(1-\cos(t)) \\ a(t - \sin(t)) \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} a(1-\cos(t)) \\ a\sin(t) \end{pmatrix}$$

and we have

$$v(\gamma(t)) \cdot \gamma'(t) = 2a^2 - 2a^2\cos(t) - a^2 + a^2\cos(t) + a^2\cos(t)$$
$$- a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t).$$

Now we can compute

$$\int_\gamma v\ d\gamma = \int_0^{2\pi} 2a^2 - 2a^2\cos(t) - a^2 + a^2\cos(t) + a^2\cos(t)$$
$$- a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t)\ dt$$
$$= \int_0^{2\pi} a^2 - a^2\cos^2(t) + a^2\sin(t)t - a^2\sin^2(t)\ dt = -2\pi a^2.$$

10.3. Magnetic field.

The following vector field describes, according to the *Biot–Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the z-axis:

$$B(x,y,z) = \frac{\mu_0 I}{2\pi}\frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x,y) \neq 0,$$

where μ_0 and I are, respectively, the magnetic constant and I the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i}B_j = \frac{\partial}{\partial x_j}B_i \quad \forall\, i,j \in \{1,2,3\},$$

where we denoted $(x_1, x_2, x_3) = (x, y, z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Is B conservative? Does B admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?

Solution.

(a) This is a direct calculation.

(b) We see that

$$\begin{aligned}\int_{\gamma_m} B \cdot d\vec{s} &= \frac{\mu_0 I}{2\pi} \int_0^{2\pi m} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt \\ &= \frac{\mu_0 I}{2\pi} \int_0^{2\pi m} dt = \mu_0 I m.\end{aligned}$$

(c) No, since the γ_m's are loops but the line integral are not zero. Note that this is not in conflict with (a) since $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is not star-shaped.

*10.4. Vector field

(1) Check that the vector-field $\mathbb{R}^2$

$$f(x, y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$$

is conservative.

(2) Compute a potential of f.

(3) Compute

$$\int_\gamma f \cdot d\vec{s},$$

where γ is the parametrised curve

$$\begin{cases} \gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \mathbb{R}^2 \\ \theta \mapsto \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(\theta), \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\theta)\right) \end{cases}$$

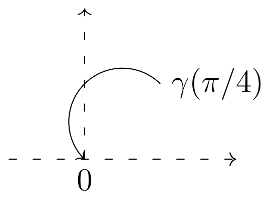

Figure 1: Curve γ.

shown in Figure 1.

Solution.

(1) One can check directly that

$$\frac{\partial}{\partial y}\left(2xy^2 - 5x^4y + 5\right) = 4xy - 5x^4 = \frac{\partial}{\partial x}\left(-7y^6 - x^5 + 2x^2y\right).$$

As $\mathbb{R}^2$ is simply connected (star-shaped), we deduce that f is conservative.

(2) A potential is given by $\omega(x, y) = x^2y^2 - x^5y + 5x - y^7$.

(3) As f is conservative, we have

$$\begin{aligned}\int_\gamma f \cdot d\vec{s} &= \varphi\left(\gamma\left(\frac{5\pi}{2}\right)\right) - \varphi\left(\gamma\left(\frac{\pi}{4}\right)\right) \\ &= \varphi(0,0) - \varphi(1,1) = -\left(1 - 1 + 5 - 1\right) \\ &= -4.\end{aligned}$$