Only the exercises with an asterisk (*) will be corrected.

11.1. MC questions.

(a) Which of the following statements are true?

Recall that, for a C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, its gradient $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ can be thought as a vector field. The equation

$$\operatorname{curl}(\nabla f) = g$$

- ☐ has a solution f for *every* given g
- ☑ has a solution f *only* for $g = 0$

  Solution. Indeed an immediate calculation shows that it is always $\operatorname{curl}(\nabla f) = 0$, hence it must be $g = 0$.

- ☐ has a solution f *also* for some nonzero g's.

(b) Which of the following statements are true?

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, regular region. Generally speaking, the integral $\int_\Omega d\mu$ represents:

- ☑ the area of Ω
- ☐ the length of the curve bounding Ω
- ☑ the volume of a cylinder with base Ω and height 1
- ☐ the surface area of some cylinder with base Ω and height 1.

Solution. Indeed the fact that the integral represents the area is well-known from the lectures. Recalling that the formula for the volume and surface area of a cylinder are, respectively,

$$(\text{Base Area}) \times (\text{height}) \quad \text{and} \quad 2(\text{Base Area}) + (\text{Base profile Length}) \times (\text{height}),$$

we also get that the third statement is true. Finally, for instance when Ω is a circle, the second and the fourth statement cannot hold.

***11.2. Integrability.**

Let $f\colon [0,1]\times[0,1]\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}1, & \text{if } (x,y)=\left(\frac{2k-1}{2^n},\frac{2l-1}{2^n}\right) \text{ for } k,\, l\in\mathbb{N} \text{ and } n\in\mathbb{N}_0,\\ 0, & \text{otherwise.}\end{cases}$$

(a) Using upper and lower sums, show that f is not integrable.

(b) Show that for all $x\in[0,1]$ the function $y\mapsto f(x,y)$ is integrable with

$$\int_0^1 f(x,y)\,dy=0$$

and therefore $\int_0^1\left(\int_0^1 f(x,y)\,dy\right)\,dx=0$.

Solution.

(a) Let

$$P_x\colon x_0=0<x_1<\cdots<x_r=1,$$
$$P_y\colon y_0=0<y_1<\cdots<y_s=1.$$

There exists an N (which depends on the partitions above), such that for every rectangle I_{ij} there exists natural numbers k,l for which

$$\left(\frac{2k-1}{2^N},\frac{2l-1}{2^N}\right)\in I_{ij}.$$

This can be seen as follows: at the middle point of the square

$$C^N_{k,l}:=\left[\frac{2(k-1)}{2^N},\frac{2k}{2^{N-1}}\right]\times\left[\frac{2(l-1)}{2^N},\frac{2l}{2^{N-1}}\right],\quad k,l\in\mathbb{N}.$$

is f by definition equal to 1. This square has the area $2^{-(N+1)}$ and it covers the entire square $[0,1]^2$. Therefore every rectangle I_{ij}, for N large enough contains this square $C^N_{k,l}$, i.e.

$$\forall i,j\,\exists k,l\in\mathbb{N}: C^N_{k,l}\subseteq I_{ij}.$$

This also shows that for every partition (P_x,P_y), the upper sum $S(P_x,P_y)$ is bounded from below by 1 — explicitly:

$$S(P_x,P_y)=\sum_i\sum_j F_{ij}\mu(I_{ij})\geq\sum_i\sum_j\mu(I_{ij})=1.$$

For the lower sum $s(P_x, P_y)$, we observe that on

$$A := ([0,1] \cap \mathbb{R} \setminus \mathbb{Q}) \times ([0,1] \cap \mathbb{R} \setminus \mathbb{Q})$$

f is equal to 0 and that A lies in $[0,1]^2$ completely. Since every square I_{ij} contains the open square $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$, A has a non-trivial intersection with I_{ij} for all pairs (i,j). It therefore holds

$$s(P_x, P_y) = 0.$$

To summarise, we have:

$$\sup_{(P_x,P_y)} s(P_x, P_y) = 0 < 1 = \sup_{(P_x,P_y)} S(P_x, P_y).$$

This means that f is not integrable.

(b)[1] For any $x \in [0,1]$ write $f_x(y) = [y \to f(x,y)]$. If $x \neq \frac{2k-1}{2^n}$ for any k, n the integral vanishes due to $f_x(y) = 0$.

For the other case, fixing x we automatically fix k_x and n_x since the representation is unique[2]. This means

$$\begin{aligned} \operatorname{supp}(f_x) &= \{y \in \mathbb{R} \mid f_x(y) \neq 0\} = \{y \in \mathbb{R} \mid f_x(y) = 1\} \\ &= \left\{ y \in \mathbb{R} \mid y = \frac{2l-1}{n_x},\ l \in \mathbb{N} \right\}. \end{aligned}$$

Now $\operatorname{supp}(f_x) \cap [0,1]$ has finite cardinality since

$$\operatorname{supp}(f_x) \cap [0,1] = \left\{ 0 \leq \frac{2l-1}{n_x} \leq 1 \mid l \in \mathbb{N} \right\} = \left\{ 1 \leq l \leq \frac{n_x+1}{2} \mid l \in \mathbb{N} \right\}$$

which in particular means that the integral of f_x over $[0,1]$ must vanish.

To see this, choose any partition $\mathcal{P} = \{I_1, \ldots, I_m\}$ of $[0,1]$ ($I_1, \ldots, I_m$ are closed intervals). Let $I_{j_1}, \ldots, I_{j_r}$ be those intervals in $\mathcal{P}$ that contain at least one point of $\operatorname{supp}(f_x)$. We find for the lower sum

$$L(f_x, \mathcal{P}) = \sum_{k=1}^{m} \inf_{y \in I_k} f_x(y) \operatorname{vol}(I_k) = \sum_{k=1}^{r} \inf_{y \in I_{j_k}} f_x(y) \operatorname{vol}(I_{j_k}) = 0$$

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[1] Thank you to Nicolas Münchinger for writing this solution!

[2] Indeed, for n, m, k, l we find (w.l.o.g. assume $n \leq m$)

$$\frac{2k-1}{2^n} = \frac{2l-1}{2^m} \iff 2^{m-n}(2k-1) = 2l-1.$$

Now, if $n < m$ we would have a contradiction since the lhs would be even and the rhs odd. Hence $n = m$ and we immediately see that also $k = l$.

as for every k there exists some $y \in I_{j_k}$ with $f_x(y) = 0$. Since $\mathcal{P}$ was arbitrary we conclude that

$$L(f_x) = \sup_{\mathcal{P}} L(f_x, \mathcal{P}) = 0.$$

For the upper sum, let $\epsilon > 0$ be arbitrary and $\operatorname{supp}(f_x) \cap [0,1] = \{s_1, \dots, s_p\}$. We cover every s_k with the interval $[s_k - \epsilon, s_k + \epsilon]$ and choose suitable intervals $I_1, \dots, I_m$ s.t.

$$\mathcal{P}_\epsilon = \{[s_k - \epsilon, s_k + \epsilon] \mid 1 \leq k \leq p\} \cup \{I_1, \dots, I_m\}$$

forms a partition of $[0,1]$. Now

$$\begin{aligned} U(f_x, \mathcal{P}_\epsilon) = \sum_{k=1}^{p} & \sup_{y \in [s_k - \epsilon, s_k + \epsilon]} f_x(y) \operatorname{vol}([s_k - \epsilon, s_k + \epsilon]) \\ & + \sum_{j=1}^{m} \sup_{y \in I_j} f_x(y) \operatorname{vol}(I_j) = 2p\epsilon. \end{aligned}$$

Since ϵ can be made arbitrarily small (p is fixed, independent of ϵ) and $U(f_x, \mathcal{P}) \geq 0$ for all partitions we infer

$$U(f) = \inf_{\mathcal{P}} U(f, \mathcal{P}) = 0.$$

Putting everything together we conclude that f_x is integrable for all $x \in [0,1]$ with

$$\int_0^1 f_x(y)\, \mathrm{d}y = U(f_x) = L(f_x) = 0.$$

*11.3. Double integrals.

Compute the following integrals:

(a) $\displaystyle\int_0^1 \int_0^x e^{x+y}\, dy\, dx$

(b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 x \cos y\, dx\, dy$

(c) $\displaystyle\int_1^3 \int_2^{5-x} \frac{1}{(x+y)^3}\, dy\, dx.$

Now describe and draw the domains of integration, and compute the integrals exchanging order of integration. Is the result the same?

Solution.

(a) We have

$$\int_0^1 \int_0^x e^{x+y}\, dy\, dx = \int_0^1 e^x \int_0^x e^y\, dy\, dx = \int_0^1 e^x(e^x - 1)\, dx = \frac{e^2}{2} - e + \frac{1}{2}.$$

The domain of integration is the triangle the enclosed by the x-axis and the lines $x = 1, x = y$. Thus we can exchange the domain of integration as follows:

$$\int_0^1 \int_y^1 e^{x+y}\, dx\, dy = \int_0^1 e^y(e - e^y)\, dy = \int_0^1 e^{y+1} - e^{2y}\, dy = [e^{y+1} - \frac{1}{2}e^{2y}]_0^1$$
$$= e^2 - \frac{e^2}{2} - e + \frac{1}{2} = \frac{e^2}{2} - e + \frac{1}{2}.$$

The result is the same.

(b) We have

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 x \cos y\, dx\, dy &= \int_0^1 \frac{1}{2}(\cos y - y \cos y)\, dy \\ &= \frac{1}{2}\left([\sin y]_0^1 - [y \sin y]_0^1 + \int_0^1 \sin x\, dx\right) \\ &= \frac{1}{2}\left([\sin y]_0^1 - [y \sin y]_0^1 - [\cos x]_0^1\right) \\ &= \frac{1}{2}(\sin 1 - \sin 1 - \cos 1 + 1) \\ &= \frac{1}{2} - \frac{\cos 1}{2}.\end{aligned}$$

The domain of integration is the region enclosed by the graphs of $x = \sqrt{y}$ and the line $x = 1$. Exchanging the order of integration yields the region between the x-axis and the parabola $y = x^2$, hence

$$\int_0^1 \int_0^{x^2} x \cos y\, dy\, dx = \int_0^1 x \sin(x^2)\, dx = -\frac{1}{2}[\cos(x^2)]_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}.$$

The result is again the same.

(c) We have:

$$\int_1^3 \int_2^{5-x} \frac{dy}{(x+y)^3} dx = \int_1^3 -\frac{1}{2}\left(\frac{1}{25} - \frac{1}{(x+2)^2}\right) dx = \frac{2}{75}.$$

The domain of integration is the triangle enclosed by the lines $y = 2$, $x = 1$ and $y = 5 - x$. We can exchange the order of integration as follows:

$$\int_2^4 \int_1^{5-y} \frac{dx}{(x+y)^3} dy = \int_2^4 -\frac{1}{2}\left(\frac{1}{25} - \frac{1}{(y+1)^2}\right) dy = \frac{2}{75}.$$

11.4. Fubini's theorem for explicit functions.

(a) Compute

$$\int_{[-1,1]\times[2,3]} \left(x^4 y - y^5 x + y^3\right) dxdy$$

(b) Let $D^2 = \mathbb{R}^2 \cap \{(x, y) : x^2 + y^2 \leq 1\}$ be the unit disk in the plane. Compute

$$\int_{D^2} x^2 y^2 dxdy$$

by following the steps below.

(I) Show that for all continuous function $f : D^2 \to \mathbb{R}$, we have

$$\int_{D^2} f(x,y)dxdy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dy\right) dx.$$

(II) Show that

$$\int_{D^2} x^2 y^2 dxdy = \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta$$

by using the formula from the previous step and by making a change of variables $x = \sin(\theta)$.

(III) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta = \frac{\pi}{32}$$

and calculate $\int_{D^2} x^2 y^2 dxdy$.

Solution.

(a) We have

$$\begin{aligned}\int_{[-1,1]\times[2,3]} \left(x^4y - y^5x + y^3\right) dxdy &= \int_2^3 \left(\int_{-1}^1 x^4y - y^5x + y^3 dx\right) dy \\ &= \int_2^3 \left(\left[y\frac{x^5}{5} - y^5\frac{x^2}{2}\right]_{-1}^1 + 2y^3\right) dxdy \\ &= \int_2^3 \left(\frac{2y}{5} + 2y^3\right) dy \\ &= \left[\frac{y^2}{5} + \frac{y^4}{2}\right]_2^3 = \frac{9-4}{5} + \frac{81-16}{2} = \frac{67}{2}.\end{aligned}$$

(b) (I) We have for all $(x, y) \in D^2$ the inequality

$$x^2 + y^2 \leq 1 \tag{1}$$

which implies that $-1 \leq x \leq 1$. Therefore, (1) holds if and only $-1 \leq x \leq 1$ and

$$y^2 \leq 1 - x^2$$

which is equivalent to (notice that $1 - x^2 \geq 0$) $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Finally, we have proved that

$$\begin{aligned}D^2 &= \mathbb{R}^2 \cap \{(x, y) : x^2 + y^2 \leq 1\} \\ &= \mathbb{R}^2 \cap \{(x, y) : -1 \leq x \leq 1 \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}.\end{aligned} \tag{2}$$

The integral formula is then a direct consequence of equation (2) and Fubini's theorem.

(II) Using the formula in part (1), we find

$$\begin{aligned}\int_{D^2} x^2y^2dxdy &= \int_{-1}^1 x^2 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2dy\right) dx = \int_{-1}^1 x^2 \left[\frac{y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \frac{2}{3}\int_{-1}^1 x^2 \left(1 - x^2\right)^{\frac{3}{2}} dx = \frac{4}{3}\int_0^1 x^2 \left(1 - x^2\right)^{\frac{3}{2}} dx\end{aligned} \tag{3}$$

where we used the symmetry of the integral in the last equality (formally, one can make a change of variable $t = -x$ in the integral $\int_{-1}^0$ to obtain the result). Now, we make the change of variable $x = \sin(\theta)$ to obtain (using $1 - \sin^2 = \cos^2$)

$$\int_0^1 x^2 \left(1 - x^2\right)^{\frac{3}{2}} dx = \int_0^{\frac{\pi}{2}} \sin^2(\theta) \left(1 - \sin^2(\theta)\right)^{\frac{3}{2}} \cos(\theta) d\theta \tag{4}$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos^4(\theta) d\theta. \tag{5}$$

(III) Using $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\cos^2 + \sin^2 = 1$, we obtain

$$\cos^4(\theta)\sin^2(\theta) = \frac{1}{4}\cos^2(\theta)\sin^2(2\theta) = \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(1 - \cos^2(2\theta)\right)$$

$$= \frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(4\theta)}{2}\right) \tag{6}$$

$$= \frac{1}{16}\left(1 + \cos(2\theta) - \cos(4\theta) - \cos(2\theta)\cos(4\theta)\right)$$

$$= \frac{1}{32}\left(2 + 2\cos(2\theta) - 2\cos(4\theta) - 2\cos(2\theta)\cos(4\theta)\right)$$

$$= \frac{1}{32}\left(2 + 2\cos(2\theta) - 2\cos(4\theta) - \cos(2\theta) - \cos(6\theta)\right)$$

$$= \frac{1}{32}\left(2 + \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)\right) \tag{7}$$

where we used

$$\cos(2\theta)\cos(4\theta) = \frac{1}{2}\left(\cos(2\theta) + \cos(6\theta)\right),$$

an identity which can be derived from the de Moivre's formula

$$\cos(2\theta)\cos(4\theta) = \frac{\left(e^{2i\theta} + e^{-2i\theta}\right)}{2}\frac{\left(e^{4i\theta} + e^{-4i\theta}\right)}{2}$$

$$= \frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta} + e^{6i\theta} + e^{-6i\theta}\right)$$

$$= \frac{1}{2}\left(\cos(2\theta) + \cos(6\theta)\right).$$

Now, we have obviously for all integer $k \geq 1$

$$\int_0^{\frac{\pi}{2}} \cos(2k\theta) d\theta = \left[\frac{\sin(2k\theta)}{k}\right]_0^{\frac{\pi}{2}} = \sin(\pi k) = 0. \tag{8}$$

Therefore, by equations (6) and (8), we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^4(\theta)\sin^2(\theta)d\theta &= \int_0^{\frac{\pi}{2}} \frac{1}{32}\left(2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)\right)d\theta \\ &= \frac{\pi}{32}.\end{aligned}$$

Remark: One could also directly expand $\cos^4(\theta)\sin^2(\theta)$ with de Moivre's formula, but the computation would be slightly longer.

We therefore obtain

$$\int_{D^2} x^2y^2dxdy = \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2(\theta)\cos^4(\theta)d\theta = \frac{4}{3}\times\frac{\pi}{32} = \frac{\pi}{24}.$$

Remark: Once we know the change of variables, we can use polar coordinates to find

$$\begin{aligned}\int_{D^2} x^2y^2dxdy &= \int_0^1\int_0^{2\pi} r^5\cos^2(\theta)\sin^2(\theta)d\theta dr = \frac{1}{6}\int_0^{2\pi}\cos^2(\theta)\sin^2(\theta)d\theta \\ &= \frac{1}{24}\int_0^{2\pi}\sin^2(2\theta)d\theta = \frac{1}{24}\int_0^{2\pi}(1-\cos^2(2\theta))d\theta \\ &= \frac{1}{24}\int_0^{2\pi}\left(1-\frac{1+\cos(4\theta)}{2}\right)d\theta = \frac{\pi}{24},\end{aligned}$$

where we used $\sin(2\theta) = \cos(\theta)\sin(\theta)$ and $\cos(2\theta) = 2\cos^2(\theta) - 1$.