Only the exercises with an asterisk (*) will be corrected.

12.1. MC questions.

(a) Choose the correct statement. Motivate your answer.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous functions and let $B_r(0) \subset \mathbb{R}^n$ be the ball of radius $r > 0$ centred ad the origin. The integral

$$\int_{B_r(0)} f(x)\,dx$$

can also be written as

☐ $r^n \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$

☐ $\frac{1}{r^n} \int_{B_1(0)} f(rx)\,dx$

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☐ $\frac{1}{r^n} \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$

where we denoted $rx = (rx_1, \ldots, rx_n)$ and similarly for $\frac{1}{r}x$.

Solution. The correct choice is the third one, namely:

$$r^n \int_{B_1(0)} f(rx)\,dx.$$

Indeed, the dilation that transforms $B_r(0)$ into $B_1(0)$ is defined by $y = \frac{1}{r}x$, hence the change of variables formula gives

$$y = rx \quad \Longrightarrow \quad dx_1 \cdots dx_n = r^n dy_1 \cdots dy_n.$$

(b) Which of the following vector fields $f\colon X \to \mathbb{R}^n$ admit a potential?

☐ $X = \mathbb{R}^2, \qquad f(x,y) = \begin{pmatrix} x \\ xy \end{pmatrix},$

Solution. Note that

$$\frac{\partial f_1}{\partial y} = 0 \neq y \frac{\partial f_2}{\partial x}.$$

The set X is open, so by Proposition 4.1.13 in the script, the vector field f is not conservative and therefore also does not admit a potential.

☐ $X = \mathbb{R}^2 \setminus \{0\}, \qquad f(x,y) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

Solution. See Example 4.1.19 (1) in the script.

☐ $X = \mathbb{R}^2, \qquad f(x,y) = \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$

Solution. Similar argument as in the first part shows that this vector field is not conservative.

☑ $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}, \qquad f(x,y,z) = \begin{pmatrix} e^z \sin(z) x \\ 0 \\ \frac{1}{2} x^2 e^z (\cos(z) + \sin(z)) \end{pmatrix}$

Solution. This vector admits a potential: $f = \nabla g$ for

$$g = \frac{1}{2} e^z x^2 \sin(z).$$

*12.2. Volume of the region enclosed by graphs of functions.

Let

$$K_1 = \left\{ (x,y,z) : 1 \le z < \infty,\ \sqrt{x^2+y^2} \le \sqrt{-1+z} \right\},$$

and

$$K_2 = \left\{ (x,y,z) : -\infty < z \le 5,\ \sqrt{x^2+y^2} \le \sqrt{5-z} \right\}.$$

Calculate the volume of $K_1 \cap K_2$.

Solution.

Let

$$N_1 = \left\{ (x,y,z) : 1 \le z \le 3,\ \sqrt{x^2+y^2} \le \sqrt{-1+z} \right\},$$
$$N_2 = \left\{ (x,y,z) : 3 \le z \le 5,\ \sqrt{x^2+y^2} \le \sqrt{5-z} \right\},$$
$$\gamma = \{ (x,y,z) : z = 3, \sqrt{x^2+y^2} = \sqrt{2} \}.$$

Then

$$K_1 \cap K_2 = N_1 \cup N_2 \setminus \gamma$$

and the volume of $K_1 \cap K_2$ is

$$V(K_1 \cap K_2) = V(N_1) + V(N_2) - V(\gamma). \tag{1}$$

Note that that the volume of γ is zero because it is 1-dimensional. We thus get $V(K_1 \cap K_2) = V(N_1) + V(N_2)$.

We calculate

$$V(N_1) = \int_1^3 \left(\int_{A_1(z)} dxdy \right) dz$$

and

$$V(N_2) = \int_3^5 \left(\int_{A_2(z)} dxdy \right) dz,$$

where

$$A_1(z) = \left\{ (x, y) : x^2 + y^2 \leq -1 + z \right\}$$

and

$$A_2(z) = \left\{ (x, y) : x^2 + y^2 \leq 5 - z \right\}.$$

Therefore

$$\begin{aligned} V(N_1) &= \int_1^3 \pi(-1 + z)\, dz \\ &= \pi \left[-z + \frac{z^2}{2} \right]_1^3 \\ &= 2\pi \end{aligned}$$

and

$$\begin{aligned} V(N_2) &= \int_3^5 \pi(5 - z)\, dz \\ &= \pi \left[5z - \frac{z^2}{2} \right]_3^5 \\ &= 2\pi. \end{aligned}$$

The volume of $K_1 \cap K_2$ is thus

$$V(K_1 \cap K_2) = 2\pi + 2\pi = 4\pi.$$

12.3. Line integral vs double integral of curl.

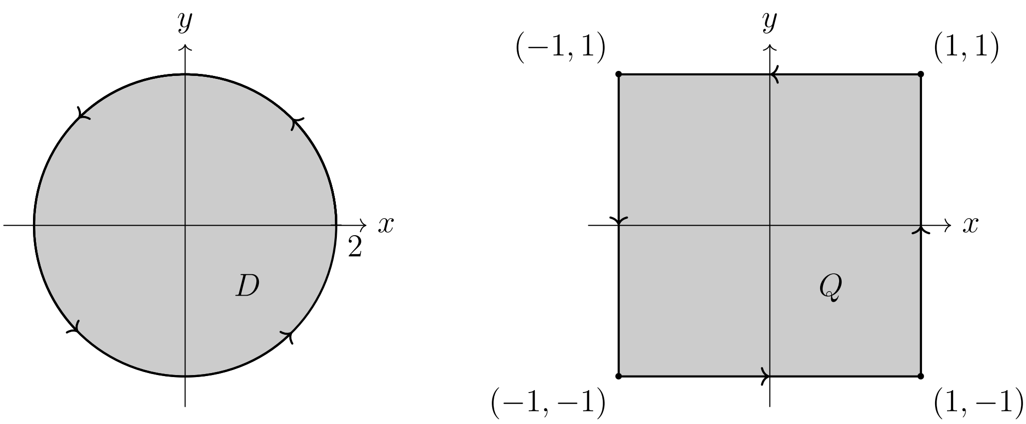
The *curl* of a vector field in $\mathbb{R}^2$ is, by definition, the function

$$\operatorname{curl}(v) = \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1.$$

Consider the vector field $v(x, y) = (y^2, x)$.

(a) Compute the line integral of v along the circle of radius 2 centered at the origin and along the square of vertices $(\pm 1, \pm 1)$, both oriented counter-clockwise (see the picture).

(b) Now compute the double integral of $\operatorname{curl}(v)$ over the disk D and the square Q enclosed by the curves in (b). What do you notice?



Solution.

(a) Parametrizing the circle ∂D with $\gamma : [0, 2\pi) \to \partial D$, $\gamma(t) = 2(\cos\varphi, \sin\varphi)$, we see that

$$\begin{aligned}\int_{\partial D} v \cdot d\vec{s} &= \int_0^{2\pi} \begin{pmatrix} 4(\sin\varphi)^2 \\ 2\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} -2\sin\varphi \\ 2\cos\varphi \end{pmatrix} d\varphi \\ &= \int_0^{2\pi} \Big(-8(\sin\varphi)^3 + 4(\cos\varphi)^2 \Big) d\varphi = 4\pi.\end{aligned}$$

As for ∂Q, parametrizations for each side are given by

$$\begin{aligned}
q_1(t) &= (1-t)(1,1) + t(-1,1) = (1-2t, 1),\\
q_2(t) &= (1-t)(-1,1) + t(-1,-1) = (-1, 1-2t),\\
q_3(t) &= (1-t)(-1,-1) + t(1,-1) = (-1+2t, -1),\\
q_3(t) &= (1-t)(1,-1) + t(1,1) = (1, -1+2t),
\end{aligned}$$

and the result is

$$\int_{\partial Q} v \cdot d\vec{s} = \sum_{j=1}^{4} \int_0^1 v(q_j(t)) \cdot q_j'(t)\, dt = 4.$$

(b) The curl of v is $\operatorname{curl}(v) = \dfrac{\partial}{\partial x} v_2 - \dfrac{\partial}{\partial y} v_1 = 1 - 2y$.

For D, using polar coordinates we compute

$$\begin{aligned}
\int_\gamma v\ d\gamma &= \int_D (1-2y) dxdy\\
&= \int_0^{2\pi} \int_0^2 (1 - 2r\sin(\varphi)) r\ drd\varphi\\
&= \int_0^{2\pi} \int_0^2 r - 2r^2 \sin(\varphi)\ drd\varphi\\
&= 2\pi \left[\frac{r^2}{2}\right]_0^2 - 2\int_0^2 r^2 dr \int_0^{2\pi} \sin(\varphi) d\varphi = 4\pi,
\end{aligned}$$

and we see that it coincides with the line integral $\int_{\partial D} v \cdot d\vec{s}$.

As for Q, we see that

$$\begin{aligned}
\int_Q (1-2y) dxdy &= \int_{-1}^1 \int_{-1}^1 (1-2y)\ dxdy\\
&= 2\int_{-1}^1 (1-2y)\ dy = 2(2 - [y^2]_{y=-1}^1) = 4.
\end{aligned}$$

Once again this coincides with $\int_{\partial Q} v \cdot d\vec{s}$

*12.4. Volume of a 3-dimensional ball.

Let $r > 0$ and $B_3(0,r) = \mathbb{R}^3 \cap \{(x,y,z) : x^2 + y^2 + z^2 < r^2\}$ be the open ball of radius $r > 0$. By using a change of coordinates, $f\colon [0,r) \times [0,2\pi) \times [0,\pi) \to B_3(0,r)$ is given as follows :

$$f(t,\theta,\varphi) = \begin{cases} t\cos(\theta)\sin(\varphi)\\ t\sin(\theta)\sin(\varphi)\\ t\cos(\varphi)\end{cases}$$

Compute the volume of $B_3(0,r)$, defined by

$$\int_{B_3(0,r)} dxdydz.$$

Solution. One immediately checks that

$$f\colon (0,r)\times[0,2\pi)\times(0,\pi)\to B_3(0,r)\setminus\{(0,0,z)\mid z\in\mathbb{R}\}$$

is a diffeomorphism, and we compute

$$Df(t,\theta,\varphi)=\begin{pmatrix} \cos(\theta)\sin(\varphi) & -t\sin(\theta)\sin(\varphi) & t\cos(\theta)\cos(\varphi)\\ \sin(\theta)\sin(\varphi) & t\cos(\theta)\sin(\varphi) & t\sin(\theta)\cos(\varphi)\\ \cos(\varphi) & 0 & -t\sin(\varphi). \end{pmatrix}$$

By expanding the last line, we find

$$\begin{aligned}\det Df(t,\theta,\varphi) &= t^2\cos(\varphi)\det\begin{pmatrix} -\sin(\theta)\sin(\varphi) & \cos(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi) & \sin(\theta)\cos(\varphi)\end{pmatrix}\\ &\quad - t^2\sin^3(\varphi)\det\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\\ &= -t^2\cos^2(\varphi)\sin(\varphi)-t^2\sin^3(\varphi)\\ &= -t^2\sin(\varphi).\end{aligned}$$

Since the set $B_3(0,r)\cap\{(0,0,z)\mid z\in\mathbb{R}\}$ is negligible, by the change of variables formula, we find (notice the absolute value)

$$\int_{B_3(0,r)} dxdydz=\int_0^r\int_0^{2\pi}\int_0^{\pi} t^2|\sin(\varphi)|dtd\theta d\varphi = 2\pi\left[\frac{t^3}{3}\right]_0^r\int_0^{\frac{\pi}{2}} 2\sin(\varphi)d\varphi=\frac{4\pi}{3}r^3.$$