Only the exercises with an asterisk (*) will be corrected.

13.1. MC questions.

(a) Which of the following statements are true?

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a non-negative, continuous function. Then

☐ If $\lim_{x\to\infty} f(x) = 0$, the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite

Solution. A counterexample already in $n = 1$ is $f(x) = \frac{1}{1+|x|}$.

☐ If the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite, then $\lim_{x\to\infty} f(x) = 0$

Solution. Consider for instance a infinite-sawtooth-like function $f : \mathbb{R} \to [0, 1]$ consisting of isosceles triangles in $[j, j + 2^{-j}]$, $j \in \mathbb{N}$ each of height 1 and 0 otherwise (see picture below): it is an elementary matter to see that $\int_{\mathbb{R}} f\,dx$ is finite but $\lim_{x\to\infty} f(x)$ does not exist.

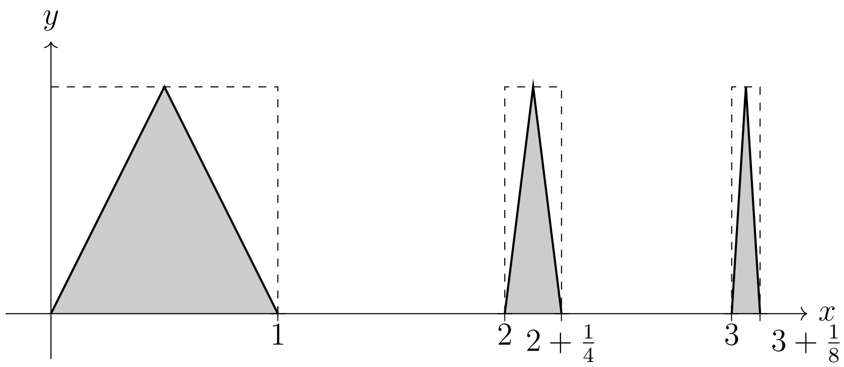

Figure 1: Sawtooth-like function

☐ If $\lim_{x\to\infty} f(x)$ does not exist, then the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite.

Solution. This is the same as the second statement.

☑ If $\lim_{x\to\infty} f(x)$ exists and is nonzero, the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite.

Solution. If $\lim_{x\to\infty} f(x) = c > 0$, then there exists $R > 0$ so that $f(x) > c/2$ if $|x| > R$, thus $\int_{\mathbb{R}^n} f\,dx \geq \int_{\mathbb{R}^n \setminus B_R} \frac{c}{2}\,dx = +\infty$.

(b) Let $D := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit ball with boundary $S := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f : D \to \mathbb{R}$ be a C^1 function, so that there exists a constant $c \neq 0$ for which

$$f(x,y) = c \quad \forall\ (x,y) \in S$$

holds. Choose the correct statement

- ☐ $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\, dxdy = c$
- ☑ $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\, dxdy = 0$
- ☐ The integral $\displaystyle\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\, dxdy$ cannot be calculated with the given information.

Solution.

We use Green's Theorem with $P = Q = f$. A parametrisation of S is given by $\gamma : [0, 2\pi] \to S$,

$$\gamma(t) = (\cos(t), \sin(t)).$$

Thus Green's Theorem gives

$$\begin{aligned}\int_D \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\, dxdy &= \int_S f(dx + dy) = \int_0^{2\pi} f(\gamma(t))(\cos'(t) + \sin'(t))\ dt \\ &= c\int_0^{2\pi} \cos(t) - \sin(t)\ dt = 0.\end{aligned}$$

*13.2. Integrals in $\mathbb{R}^2$.

Calculate the following integrals.

(a) $\displaystyle\int_D xy(x^2 + y^2)\, \mathrm{d}(x,y), \qquad D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 - 2x < 0\}.$

(b) $\displaystyle\int_D \frac{\sin(x^2 + y^2)}{2 + \cos(x^2 + y^2)}\, \mathrm{d}(x,y), \qquad D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 < x^2 + y^2 < 4\}.$

(c) $\displaystyle\int_D \frac{1}{(x^2 + y^2)^2}\, \mathrm{d}(x,y), \qquad D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1, x + y > 1\}.$

Solution.

(a) We first note that

$$\begin{aligned} &0 > x^2 + y^2 - 2x = x^2 - 2x + 1 - 1 + y^2 = (x-1)^2 + y^2 - 1 \\ \iff\ &1 > (x-1)^2 + y^2 \end{aligned}$$

holds for all $(x, y) \in \mathbb{R}^2$. From there, it follows that D is a circle with radius 1 around the point $(1, 0)$. We parametrise D by using the polar coordinates $\Phi\colon (0,1) \times (-\pi, \pi) \to \mathbb{R}^2$,

$$\Phi(r, \varphi) = (1 + r\cos(\varphi), r\sin(\varphi)),$$

where $D = \Phi(D') \cup N$ for

$$D' = \{(r, \varphi) \in (0,1) \times (0, 2\pi)\}$$

and $N = \{(x, 0) \,|\, 0 \leq x < 1\}$ is a negligible set. We calculate

$$\begin{aligned} &\int_D xy(x^2+y^2)\,\mathrm{d}(x,y) \\ &= \int_0^1 \int_{-\pi}^{\pi} (1 + r\cos(\varphi))r\sin(\varphi)((1+r\cos(\varphi))^2 + r^2\sin(\varphi)^2)r\,d\varphi\,dr \\ &= 0 \end{aligned}$$

where we used that

$$f_r(\varphi) = (1 + r\cos(\varphi))r\sin(\varphi)((1+r\cos(\varphi))^2 + r^2\sin(\varphi)^2)r$$

is an odd function for every r: $f_r(-\varphi) = -f_r(\varphi)$ (since sin is an odd and cos an even function), and therefore the integral over $(-\pi, \pi)$ vanishes.

(b) By using the polar coordinates, we obtain

$$\begin{aligned} \int_D \frac{\sin(x^2+y^2)}{2+\cos(x^2+y^2)}\,\mathrm{d}(x,y) &= \int_0^{2\pi}\int_1^2 \frac{\sin(r^2)}{2+\cos(r^2)} r\,dr\,d\varphi \\ &= \pi \int_1^2 \frac{2r\sin(r^2)}{2+\cos(r^2)}\,dr \\ &= \pi \Big[-\log(2+\cos(r^2))\Big]_{r=1}^2 \\ &= \pi \log\left(\frac{2+\cos 1}{2+\cos 4}\right). \end{aligned}$$

(c) In polar coordinates $\Psi : D' \to D$ we can parametrise D by

$$D' = \left\{(r, \varphi) \in (0, \infty) \times \left(0, \frac{\pi}{2}\right) \,\middle|\, (\cos\varphi + \sin\varphi)^{-1} < r < 1\right\}.$$

We obtain:

$$\int_D \frac{1}{(x^2+y^2)^2}\,\mathrm{d}(x,y) = \int_0^{\frac{\pi}{2}} \int_{(\cos\varphi+\sin\varphi)^{-1}}^1 \frac{1}{r^4} r\,dr\,d\varphi$$
$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-(\cos\varphi+\sin\varphi)^2)\,d\varphi$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\varphi)\,d\varphi = \frac{1}{2}.$$

13.3. The Astroid.

Let $a > 0$. The Astroid $A(a) \subset \mathbb{R}^2$ is the geometric figure in the plane defined by

$$A(a) := \left\{(x,y)\in\mathbb{R}^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right\}$$

The construction of an Astroid is very geometric (see here). Let $B(a)$ denote the set

$$B(a) = \{(rx, ry) \subset \mathbb{R}^2 \mid r\in[0,1],\ (x,y)\in A(a)\}.$$

Compute the area of $B(a)$ using the theorem of Green. Note that the theorem of Green is indeed applicable in this case.

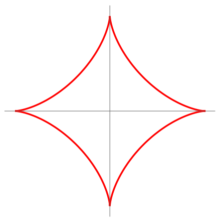

Figure 2: The red Astroid $A(a)$ is the boundary of $B(a)$.

Solution. A parametrisation of $A(a)$ is such that $\gamma : [0,2\pi]\to\mathbb{R}^2$,

$$\gamma(t) = (\gamma_1(t),\gamma_2(t)) = (a\cos^3(t), a\sin^3(t)).$$

Thanks to Green's theorem, we have

$$\text{Area}(B(a)) = \iint_{B(a)} 1\ dxdy = \int_{A(a)} (0,x)\cdot d\vec{s}$$
$$= \int_0^{2\pi} \gamma_1(t)\dot\gamma_2(t)\ dt$$
$$= 3a^2\int_0^{2\pi} \cos^4(t)\sin^2(t)\ dt$$

Now, compute that

$$\cos^4(\theta)\sin^2(\theta) = \frac{1}{32}\left(2 + \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)\right)$$

and as for all integer $k \geq 1$, we have

$$\int_0^{2\pi} \cos(k\theta)d\theta = 0,$$

we deduce that

$$\int_0^{2\pi} \cos^4(t)\sin^2(t)\ dt = \frac{\pi}{8},$$

so that

$$\text{Area}(B(a)) = \frac{3\pi a^2}{8}.$$

***13.4. Improper integral.**

Show that the limit

$$\lim_{R\to\infty} \int_{[0,R]^2} \sin(x^2+y^2)\, d(x,y)$$

exists.

Solution.

Since the integrand is continuous, we can use Fubini's Theorem and write:

$$\int_{[0,R]^2} \sin(x^2+y^2)\, d(x,y) = \int_0^R \int_0^R \sin(x^2+y^2)\, dx\, dy, \quad \forall R \geq 0.$$

By using the summation formula $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, we obtain:

$$\begin{aligned}\int_0^R \int_0^R \sin(x^2+y^2)\, dx\, dy = &\int_0^R \cos(y^2) \int_0^R \sin(x^2)\, dx\, dy \\ &+ \int_0^R \sin(y^2) \int_0^R \cos(x^2)\, dx\, dy.\end{aligned}$$

We rewrite $\int_0^R \sin(x^2)\, dx$ by using the substitution $x^2 = t$:

$$\int_0^R \sin(x^2)\,dx = \int_0^{R^2} \frac{\sin(t)}{2\sqrt{t}}\,dt.$$

Let

$$f\colon \mathbb{R}_{\geq 0} \to [-1,1], \quad f(R) := \int_0^{R^2} \frac{\sin(t)}{2\sqrt{t}}\,dt$$

and analogously

$$g(R) := \int_0^{R^2} \frac{\cos(t)}{2\sqrt{t}}\,dt.$$

We have thus shown:

$$\int_{[0,R]^2} \sin(x^2+y^2)\,d(x,y) = f(R)\cdot\int_0^R \cos(y^2)\,dy + g(R)\cdot\int_0^R \sin(y^2)\,dy.$$

By repeating the previous step, we get:

$$\int_{[0,R]^2} \sin(x^2+y^2)\,d(x,y) = f(R)\cdot g(R) + g(R)\cdot f(R) = 2f(R)\cdot g(R).$$

It suffices therefore to show that $f(R)$ and $g(R)$ converge for $R\to\infty$. We show this in the case of f (the proof for g is analogous). We have

$$\lim_{R\to\infty} 2f(R) = \int_0^{\pi/2} \sin(t)t^{-1/2}\,dt + \lim_{R\to\infty}\int_{\pi/2}^{R^2} \sin(t)t^{-1/2}\,dt.$$

The left integral exists because

$$\int_0^{\pi/2} |\sin(t)t^{-1/2}|\,dt \leq \int_0^{\pi/2} t^{-1/2}\,dt = [2\cdot t^{1/2}]_{t=0}^{t=\pi/2} \in \mathbb{R},$$

cf. Lemma 5.51 from the Analysis I script. We need to use Lemma 5.51 here because the integrand does not exist at 0 and therefore the integral must be regarded as improper.

We calculate the right-hand side by using partial integration:

$$\begin{aligned}\int_{\pi/2}^{+\infty} \sin(t)\cdot t^{-1/2}\,dt &= \left[\frac{-\cos(t)}{\sqrt{t}}\right]_{t=\pi/2}^{t=\infty} - \int_{\pi/2}^{+\infty} \frac{\cos(t)}{2t^{3/2}}\,dt \\ &= -\int_{\pi/2}^{+\infty} \frac{\cos(t)}{2t^{3/2}}\,dt.\end{aligned}$$

But

$$\left|\frac{\cos(t)}{t^{3/2}}\right| \leq \frac{1}{t^{3/2}},$$

where the latter is integrable from $\pi/2$ to $+\infty$ (with indefinite integral $-2 \cdot t^{-1/2}$). Therefore Lemma 5.51 from Analysis I again implies that $\int_{\pi/2}^{+\infty} \sin(t) \cdot t^{-1/2}\, dt$ exists.

We have therefore shown that $2f(R)$ converges for $R \to +\infty$, which also implies the convergence of $f(R)$. This completes the proof.

13.5. Fubini's theorem for an explicit integral.

By computing the integral (justify why it converges)

$$I = \int_{[0,\infty)\times[a,b]} e^{-xy} dxdy$$

in two different ways, where $0 < a < b$, compute

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

Solution. As $0 < a \leq y \leq b$, and $e^{-xy} \geq 0$, we observe that for all $R > 0$

$$0 \leq \int_{[0,R]\times[a,b]} e^{-xy} dxdy \leq (b-a) \int_0^R e^{-ax} dx = \left(\frac{b}{a} - 1\right)(1 - e^{-aR}) \leq \frac{b}{a} - 1.$$

so the integral converges. Therefore, we have

$$I = \lim_{R\to\infty} \int_{[0,R]\times[a,b]} e^{-xy} dxdy = \lim_{R\to\infty} I(R),$$

and we have by Fubini's theorem

$$I(R) = \int_a^b \left(\int_0^R e^{-xy} dx\right) dy = \int_a^b \left(\frac{1}{y} - \frac{e^{-Ry}}{y}\right) dy = \log\left(\frac{b}{a}\right) - \int_a^b \frac{e^{-Ry}}{y} dy$$

Furthermore, we have (as $a > 0$)

$$\left|\int_a^b \frac{e^{-Ry}}{y} dy\right| = \int_a^b \frac{e^{-Ry}}{y} dy \leq \frac{e^{-Ra}}{a} \int_a^b dy = \left(\frac{b}{a} - 1\right) e^{-Ra} \xrightarrow{R\to\infty} 0$$

so that

$$I = \log\left(\frac{b}{a}\right).$$

Now, by Fubini theorem, we also have (by definition of the 'improper' integral)

$$I(R) = \int_0^R \left(\int_a^b e^{-xy} dy \right) dx = \int_0^R \frac{e^{-ax} - e^{-bx}}{x} dx \xrightarrow{R \to \infty} \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

Finally, we deduce that

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \log \left(\frac{b}{a} \right).$$

Remark: Notice that the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$$

converges absolutely as $\frac{e^{-ax} - e^{-bx}}{x} = (b - a) + O(|x|)$ as $x \to 0$, and the convergence at infinity is trivial with the two exponential functions.