14.1. MC questions.

(a) Which of the following is the tangent plane of the ellipsoid:

$$2x^2 + 2y^2 + \frac{1}{4}z^2 = 1$$

which is parallel to the plane $x + y + z = 1$?

- ☐ $x + y + z = 0$
- ☐ $x + y + z = k$ for $k \in \left\{\pm\frac{2}{\sqrt{5}}\right\}$
- ☑ $x + y + z = k$ for $k \in \left\{\pm\sqrt{5}\right\}$
- ☐ $x + y + z = k$ for $k \in \{\pm 1\}$

Solution. We look for the points on the ellipsoid for which the gradient is parallel to the normal vector $(1, 1, 1)$ of the plane. We let

$$f(x, y, z) = 2x^2 + 2y^2 + \frac{z^2}{4}$$

and thus we must have

$$\nabla f\,(x, y, z) = \left(4x, 4y, \frac{z}{2}\right) = a(1, 1, 1)$$

for a real number a. It follows that

$$x = y = \frac{a}{4}, \qquad z = 2a.$$

Substituting into the equation for the ellipsoid, we obtain

$$1 = f(x, y, z) = \frac{a^2}{8} + \frac{a^2}{8} + a^2 = \frac{5}{4}a^2 \implies a_\pm = \pm\frac{2}{\sqrt{5}}.$$

In order to be parallel to the plane $x + y + z = 1$, the tangent plane has to satisfy the equation $x + y + z = b$ for $b \in \mathbb{R}$. Since

$$(x, y, z) = \left(\frac{a_+}{4}, \frac{a_+}{4}, 2a_+\right) = \left(\frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$$

and

$$(x, y, z) = \left(\frac{a_-}{4}, \frac{a_-}{4}, 2a_-\right) = \left(-\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$$

must be satisfied, we find that $b_\pm = \pm\sqrt{5}$. The tangent planes are therefore

$$x + y + z = \pm\sqrt{5}.$$

(b) For which functions $\phi, \psi, \chi \colon \{(x, y, z) \in \mathbb{R}^3 \mid y, z \neq 0\} \to \mathbb{R}$ there exists a function f such that $f_x = \phi$, $f_y = \psi$, and $f_z = \chi$?

☑ $\phi(x, y, z) = \frac{2x}{y},\ \psi(x, y, z) = 3y^2z^2 - \frac{x^2}{y^2},\ \chi(x, y, z) = 2y^3z.$

Solution. Such an f exists: $f(x, y, z) = \frac{x^2}{y} + y^3z^2 + C$.

☐ $\phi(x, y, z) = e^y + 2xy^3z^2,\ \psi(x, y, z) = xe^y + 3x^2y^2z^2,\ \chi(x, y, z) = 2x^2y^3 + x.$

Solution. Since $\psi_z \neq \chi_y$, there does not exist such an f.

☐ $\phi(x, y, z) = e^z y \cos(xy),\ \psi(x, y, z) = e^z x \cos(xy),\ \chi(x, y, z) = e^z \sin(xy).$

Solution. Such an f exists: $f(x, y, z) = e^z \sin(xy) + C$.

☐ $\phi(x, y, z) = ze^x,\ \psi(x, y, z) = \frac{1}{z}\sin(\frac{y}{z}),\ \chi(x, y, z) = \frac{y}{z^2}\sin(\frac{y}{z}) + e^x.$

Solution. Since $\psi_z \neq \chi_y$, there does not exist such an f.

(c) For which of the following pairs of functions $\phi, \psi \colon \mathbb{R}^2 \to \mathbb{R}$ does there exist a function $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $f_x(x, y) = \phi(x, y)$ and $f_y(x, y) = \psi(x, y)$?

☑ $\phi(x, y) = \left(\frac{y^3}{3}\right)\sinh(x) + y + \frac{x^2}{2}$ and $\psi(x, y) = y^2\cosh(x) + x$.

☐ $\phi(x, y) = x\sin(xy)$ and $\psi(x, y) = x\sin(xy) + 3x$.

☑ $\phi(x, y) = e^{x+\sin y}$ and $\psi(x, y) = \cos(y)e^{x+\sin(y)}$.

☑ $\phi(x, y) = xye^{x^2y}$ and $\psi(x, y) = \frac{1}{2}x^2e^{x^2y}$.

☐ $\phi(x, y) = \sinh(x^2y)$ and $\psi(x, y) = \sinh(xy^2)$.

Solution. We know that such a function f exists if and only if

$$\phi_y(x, y) = \psi_x(x, y).$$

Checking this identity for the given pairs, we see that f exists only in the first, third, and fourth case.

14.2. Extrema

Find the extrema of the function

$$f \colon D \to \mathbb{R}, \qquad f(x, y) = \exp(3y^2 - 1 - x^2)$$

for

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 4\right\}.$$

Solution. The extrema in the interior of D are the solutions of $\nabla f(x,y) = 0$, i.e. $(x,y) = (0,0)$.

To find the extrema on the boundary ∂D, we first parametrise ∂D:

$$\gamma(t) = \left(2\cos t, \frac{2\sin t}{t}\right) = \left(2\cos t, \sqrt{2}\sin t\right).$$

Then

$$f|_{\partial D} = \exp(6\sin^2 t - 4\cos^2 t - 1).$$

The extrema on the boundary satisfy $\frac{d}{dt} f(\gamma(t)) = 0$, i.e.

$$\frac{d}{dt}\exp(6\sin^2 t - 4\cos^t - 1) = \exp(6\sin^2 t - 4\cos^2 t - 1)\cdot 10 \cdot \sin 2t = 0.$$

The solutions to this equation are $t \in \{0, \pi/2, \pi, 3\pi/2\}$. The candidates for the extrema are:

	P_1	P_2	P_3	P_4	P_5
(x,y)	$(0,0)$	$(2,0)$	$(0,\sqrt{2})$	$(-2,0)$	$(0,-\sqrt{2})$
$f(x,y)$	$\frac{1}{e}$	$\frac{1}{e^5}$	e^5	$\frac{1}{e^5}$	e^5

The minimum is $\frac{1}{e^5}$ and is attained at the points $(\pm 2, 0)$. The maximum is e^5 and is attained at the points $(0, \pm\sqrt{2})$.

14.3. Volume

Calculate the volume between the ellipse $x^2 + 4y^2 \leq 1$ and the surface $z = 1 - x^2$.

Hint: Find the coordinates in the xy-plane in which the ellipse has a simpler form.

Solution. We obtain the wanted volume by integrating $f(x,y) = 1 - x^2$ over the surface $A \subset \mathbb{R}^2$ that is bounded by the ellipse $x^2 + 4y^2 = 1$:

$$V = \iint_A f(x,y)\,dxdy.$$

With the following coordinate change

$$(x,y) \to (s,\varphi), \qquad (x,y) \mapsto \left(s\cos\varphi, \frac{s}{2}\sin\varphi\right)$$

we simplify the problem. The ellipse is given by the equation $s = 1$. The function f is given by $f(s, \varphi) = 1 - s^2 \cos^2(\varphi)$. We thus get

$$\begin{aligned}\iint_A f(x,y)dxdy &= \int_0^1 \int_0^{2\pi} f(s,\varphi)\frac{s}{2}d\varphi ds \\ &= \frac{1}{2}\int_0^1 \int_0^{2\pi} (s - s^3\cos^2\varphi)d\varphi ds \\ &= \frac{1}{2}\int_0^1 (2\pi s - \pi s^3)ds \\ &= \frac{\pi}{2}\left(s^2 - \frac{s^4}{4}\right)\Big|_0^1 = \frac{3\pi}{8}.\end{aligned}$$

14.4. More volume

Let D be a surface bounded by the straight line from $(0,0)$ to $(1,0)$ and the arc parametrised by $\rho = \sin\left(\frac{\varphi}{4}\right)$, $0 \leq \varphi \leq 2\pi$. Calculate the volume of unit ball

$$\{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\}$$

that lies over the surface D.

Solution. The surface D is given by

$$B = \left\{(\rho,\varphi) \mid 0 \leq \rho \leq \sin\left(\frac{\varphi}{4}\right)\ 0 \leq \varphi < 2\pi\right\}.$$

The required volume is then

$$\begin{aligned}V &= \int_D \sqrt{1-\rho^2}\,dF \qquad \text{with } dF = \rho d\rho d\varphi \\ &= \int_0^{2\pi}\int_0^{\sin(\varphi/4)} \sqrt{1-\rho^2}\,\rho d\rho d\varphi \\ &= \int_0^{2\pi}\left[-\frac{1}{2}\frac{2}{3}(1-\rho^2)^{3/2}\right]_0^{\sin(\varphi/4)} d\varphi \\ &= -\frac{1}{3}\int_0^{2\pi}\left(\cos^3\left(\frac{\varphi}{4}\right) - 1\right)d\varphi \\ &= -\frac{1}{3}\int_0^{2\pi}\left(\frac{1}{4}\cos\left(\frac{3\varphi}{4}\right) + \frac{3}{4}\cos\left(\frac{\varphi}{4}\right) - 1\right)d\varphi \\ &= -\frac{1}{3}\left[\frac{1}{3}\sin\left(\frac{3\varphi}{4}\right) + 3\sin\left(\frac{\varphi}{4}\right) - \varphi\right]_0^{2\pi} \\ &= -\frac{1}{3}\left(-\frac{1}{3} + 3 - 2\pi\right) = \frac{2\pi}{3} - \frac{8}{9}.\end{aligned}$$

14.5. Intersection point

The equation $z = 2y^2 + x^2$ describes a surface S in $\mathbb{R}^3$, which contains the point $P = (1, 1, 3)$. Find the coordinates of the other point of S that lies on the normal to S at P.

Solution. Let $f(x, y, z) = x^2 + 2y^2 - z$. The surface S is then the level set $f(x, y, z) = 0$. The normal at the point (x, y, z) is given by

$$\nabla f = (2x, 4y, -1).$$

The normal direction at P is thus given by $n(P) = (2, 4, -1)$. The normal line through P is thus

$$g\colon \mathbb{R} \to \mathbb{R}^3, \qquad g(t) = (1, 1, 3) + t(2, 4, -1).$$

To find the other point that lies in $\operatorname{im}(g) \cap S$, we need to solve $f(g(t)) = 0$:

$$(1 + 2t)^2 + 2(1 + 4t)^2 - 3 + t = 21t + 36t^2 = 0.$$

We get $t = 0$ or $t = -\frac{21}{36} = -\frac{7}{12}$. For $t = 0$ we get P and the other intersection point is

$$P' = \left(-\frac{1}{6}, -\frac{4}{3}, \frac{43}{12}\right).$$

14.6. Global extrema

Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 3x^2 - 2xy + 3y^2 - 4x - 4y + 4.$$

Find the global extrema of f on

$$B = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Solution. We first calculate the partial derivatives of f:

$$f_x = 6x - 2y - 4, \qquad f_y = -2x + 6y - 4.$$

At an extremal point, $f_x = f_y = 0$ holds. This gives us a linear system of equations:

$$6x - 2y = 4, \qquad -2x + 6y = 4$$

with the solution $x = 1, y = 1$. Since $(1,1)$ is not contained in B, we have to try to find extrema on the boundary ∂B. We parametrise the boundary by

$$s(\phi) = (\cos(\phi), \sin(\phi)).$$

Substituting and using $\sin^2(\phi) + \cos^2(\phi) = 1$ gives

$$\begin{aligned}(f \circ s)(\phi) &= 3\cos^2\phi - 2\cos\phi\sin\phi + 3\sin^2\phi - 4\cos\phi - 4\sin\phi + 4\\ &= 3 - 2\cos\phi\sin\phi - 4\cos\phi - 4\sin\phi + 4.\end{aligned}$$

Solving the equation

$$\frac{d}{d\phi} f \circ s = -2(\cos\phi - \sin\phi) \circ (\cos\phi + \sin\phi + 2) = 0,$$

we obtain $\phi = \pi/4$ and $\phi = 5\pi/4$. Finally, we get

$$p_{\min} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad f(p_{\min}) = 6 - 4\sqrt{2}$$

and

$$p_{\max} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \qquad f(p_{\max}) = 6 + 4\sqrt{2}.$$

14.7. Initial value problem Find the solution of the initial value problem:

$$u'' + u = F(t), \qquad u(0) = u'(0) = 0,$$

where F_0 is a constant and

$$F(t) = \begin{cases} F_0 t, & 0 \le t < \pi \\ F_0(2\pi - t), & \pi \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases}$$

Solution. Since F is defined piecewise, to find the general solution of this differential equation, we have to consider each part of F separately.

The solution of the homogeneous equation is

$$u_h(t) = A\cos t + B\sin t$$

(since the characteristic polynomial $\lambda^2 + 1$ has two zeros $\lambda = \pm i$). For the particular solution we have:

- In $(0, \pi)$: with the Ansatz $u_p(t) = \alpha t + \beta$, we can check that $u_p(t) = F_0 t$ is a solution.
- In $(\pi, 2\pi)$: analogously holds $u_p(t) = F_0(2\pi - t)$.
- In $(2\pi, \infty)$ the equation is homogeneous.

The general solution of the equation must thus be of the form

$$u(t) = \begin{cases} A_1 \cos t + B_1 \sin t + F_0 t, & \text{if } 0 < t < \pi \\ A_2 \cos t + B_2 \sin t + F_0(2\pi - t), & \text{if } \pi < t < 2\pi \\ A_3 \cos t + B_3 \sin t, & \text{if } t > 2\pi. \end{cases}$$

where A_i, B_i are constants. Since $u\colon [0, \infty) \to \mathbb{R}$ solves the equation, it must be at least twice differentiable. In particular, u and the derivative u' are continuous functions. We can use this fact to determine the constants A_2, B_2, A_3, B_3 as functions of A_1 and B_1:

- First we calculate the derivative u':

$$u'(t) = \begin{cases} -A_1 \sin t + B_1 \cos t + F_0, & \text{if } 0 < t < \pi \\ -A_2 \sin t + B_2 \cos t - F_0, & \text{if } \pi < t < 2\pi \\ -A_3 \sin t + B_3 \cos t, & \text{if } t > 2\pi. \end{cases}$$

- From the continuity of u and u' at $t = \pi$ it follows:

$$\begin{aligned} -A_1 + F_0\pi &= \lim_{t\to\pi^-} u(t) = \lim_{t\to\pi^+} u(t) = -A_2 + F_0\pi \\ -B + F_0 &= \lim_{t\to\pi^-} u'(t) = \lim_{t\to\pi^+} u'(t) = -B_2 - F_0. \end{aligned}$$

  Therefore $A_1 = A_2$ and $B_2 = B_1 - 2F_0$.

- From the continuity of u and u' at $t = 2\pi$ it follows:

$$\begin{aligned} A_2 &= \lim_{t\to 2\pi^-} u(t) = \lim_{t\to 2\pi^+} u(t) = A_3, \\ B_2 - F_0 &== \lim_{t\to 2\pi^-} u'(t) = \lim_{t\to 2\pi^+} u'(t) = B_3. \end{aligned}$$

  Therefore $A_3 = A_2 = A_1$ and $B_3 = B_2 - F_0 = B_1 - 3F_0$.

We therefore obtain

$$u(t) = \begin{cases} A_1 \cos t + B_1 \sin t + F_0 t, & \text{if } 0 < t < \pi \\ A_1 \cos t + (B_1 - 2F_0) \sin t + F_0(2\pi - t), & \text{if } \pi \le t < 2\pi \\ A_1 \cos t + (B_1 - 3F_0) \sin t, & \text{if } t \ge 2\pi. \end{cases}$$

Note that by continuity of u, the values $u(\pi) := \lim_{t\to\pi^-} u(t) = \lim_{t\to\pi^+} u(t)$ (and analogously for $u(2\pi)$) are well defined.

Now from the initial conditions, we get:

$$\begin{aligned} 0 &= u(0) = \lim_{t\to 0^+} u(t) = A_1, \\ 0 &= u'(0) = \lim_{t\to 0^+} u'(t) = B_1 + F_0. \end{aligned}$$

The solution of the initial value problem is thus

$$u(t) = \begin{cases} -F_0 \sin t + F_0 t, & \text{if } 0 \le t < \pi \\ -3F_0 \sin t + F_0(2\pi - t), & \text{if } \pi \le t < 2\pi \\ -4F_0 \sin t, & \text{if } t \ge 2\pi. \end{cases}$$

14.8. Tangent planes

Let f be any differentiable function of one variable. Show that all tangent planes of the surface

$$z = y \cdot f\left(\frac{x}{y}\right)$$

pass through the point $(0, 0, 0)$.

Solution.

Let

$$G(x, y) := y \cdot f\left(\frac{x}{y}\right).$$

Then the surface $z = y \cdot f\left(\frac{x}{y}\right)$ is equal to the graph of G. It holds

$$G_x(x, y) = f'\left(\frac{x}{y}\right), \qquad G_y(x, y) = f\left(\frac{x}{y}\right) - \frac{x}{y} f'\left(\frac{x}{y}\right).$$

The tangent planes in the points $(x_0, y_0, G(x_0, y_0))$ (with $y_0 \neq 0$) is

$$\begin{aligned} z &= G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) \\ &= x f'\left(\frac{x_0}{y_0}\right) + y\left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right)\right]. \end{aligned}$$

The point $(0,0,0)$ satisfies this equation, which means that $(0,0,0)$ lies on this tangent plane. In other words, all tangent planes pass though the origin.

14.9. Global extrema

Let

$$f(x,y) \coloneqq xy(2x-5y)$$

be a function defined on the closed square with corners at $(0,0),(0,2),(2,2),(2,0)$. Find the global extrema of f.

Solution. Since the function f is differentiable everywhere, we need to consider the points on the boundary of the domain, as well as the points in the interior in which the derivative of f vanishes.

In the interior, we have

$$f_x(x,y) = 4xy - 5y^2, \qquad f_y(x,y) = 2x^2 - 10xy.$$

From $f_x(x,y) = f_y(x,y) = 0$ it follows that $x = y = 0$, but $(0,0) \notin (0,2) \times (0,2)$. Therefore there are no extrema in the interior.

As for the boundary, we consider the points corners and the rest of the points separately.

- If $(x,y) \in \{0\} \times (0,2)$, then $f(0,y) = 0$ is constant. Therefore all the points in this part of the boundary are candidates for the extrema.
- If $(x,y) \in \{2\} \times (0,2)$, then $f(2,y) = 8y - 10y^2$. By setting the derivative of $y \mapsto f(2,y)$ to zero, we get the candidate point $(2, \frac{2}{5})$ with $f(2, \frac{2}{5}) = \frac{8}{5}$.
- If $(x,y) \in (0,2) \times \{0\}$, then $f(x,0) = 0$ is constant.
- If $(x,y) \in (0,2) \times \{2\}$, then $f(x,2) = 4x^2 - 20x$. After setting the derivative of this function to zero, we get the point $(\frac{5}{2}, 2)$. This point however does not belong to the domain of definition of f.
- For the corners, we have: $f(0,0) = f(0,2) = f(2,0) = 0$ and $f(2,2) = -24$.

After comparing all the candidate points, we conclude that the global maximum is $\frac{8}{5}$, attained at the point $(2, \frac{2}{5})$ and that the global minimum is -24, attained at $(2,2)$.