0.1.

(a) Determine the solutions to the differential equation

$$y''' - 4y'' + 6y' = 0$$

satisfying the conditions

$$\lim_{t\to-\infty} y(t) = 0 \quad \text{and} \quad y'(0) = 0.$$

(b) Determine a solution of the equation

$$y''' - 4y'' + 6y' = e^{2t} + 9t^2.$$

Hint: since the equation is linear, you may divide the right-hand side in 2 parts and then sum the solutions (superposition principle).

(c) Determine the solution of the problem

$$\begin{cases} y = 2t^2 y' & \text{for } t \geq 1, \\ y(1) = 1. \end{cases}$$

Solution.

(a) The characteristic polynomial of the equation is,

$$\lambda^3 - 4\lambda^2 + 6\lambda = 0,$$

and its roots are then

$$\lambda_1 = 0, \quad \lambda_2 = 2 + i\sqrt{2}, \quad \lambda_3 = 2 - i\sqrt{2}.$$

The general solution is then

$$y(t) = C_1 + e^{2t}(C_2 \sin(\sqrt{2}t) + C_3 \cos(\sqrt{2}t)), \quad C_1, C_2, C_3 \in \mathbb{R}.$$

Since $\lim_{t\to-\infty} y(t) = C_1$, the first condition implies $C_1 = 0$. Next, since

$$y'(t) = 2e^{2t}(C_2 \sin(\sqrt{2}t) + C_3 \cos(\sqrt{2}t)) + \sqrt{2}e^{2t}(C_2 \cos(\sqrt{2}t) - C_3 \sin(\sqrt{2}t)),$$

The second condition implies $C_2 + C_3\sqrt{2} = 0$, that is $C_2 = -\sqrt{2}C_3$. So the solutions with the required properties are all of the form

$$y(t) = Ce^{2t}(-\sqrt{2}\sin(\sqrt{2}t) + \cos(\sqrt{2}t)), \quad C \in \mathbb{R}.$$

(b) The solution of the homogeneous equation is known from (a). Following the hint, we first solve the inhomogenous equations for each term separately.

- Ansatz for e^t: $y_1(t) = ae^{2t}$
  $\Longrightarrow y_1' = 2ae^{2t}, \quad y_1'' = 4ae^{2t}, \quad y_1''' = 8ae^{2t}$. Substitution
  $$(8a - 16a + 12a)e^{2t} \overset{!}{=} e^{2t}.$$
  $\Rightarrow a = \frac{1}{4} \Rightarrow y_1(t) = \frac{1}{4}e^{2t}$.
- Ansatz für $9t^2$: $\lambda = 0$ iis a zero of the characteristic polynomial.
  $\Rightarrow y_2(t) = at^3 + bt^2 + ct$.
  $\Rightarrow y_2'(t) = 3at^2 + 2bt + c$, $\Rightarrow y_2''(t) = 6at + 2b$ und $\Rightarrow y_2'''(t) = 6a$.
  Substitution:
  $$\begin{aligned} 6a - 4(6at + 2b) + 6(3at^2 + 2bt + c) &\overset{!}{=} 9t^2 \\ 18at^2 + t(-24a + 12b) + 6a - 8b + 6c &\overset{!}{=} 9t^2. \end{aligned}$$
  Koeffizientenvergleich: $18a = 9 \quad \Rightarrow \quad a = \frac{1}{2}$.
  $-24a + 12b = 0 \quad \Rightarrow \quad b = 2a = 1$.
  $6a - 8b + 6c = 0 \quad \Rightarrow \quad 6c = 8b - 6a = 8 - 3 = 5 \Rightarrow c = \frac{5}{6}$.
  $\Longrightarrow y_2(t) = \frac{1}{2}t^3 + t^2 + \frac{5}{6}t$.

Thus, since the equation is linear, the superposition principle gives

$$y_p(t) = y_1(t) + y_2(t) = \frac{1}{4}e^{2t} + \frac{1}{2}t^3 + t^2 + \frac{5}{6}t.$$

(c) The equation is separable:

$$\begin{aligned} y &= 2t^2 y'; \\ \frac{y'}{y} &= \frac{1}{2t^2}; \\ \log(|y|) &= -\frac{1}{2t} + c; \\ |y| &= e^{-\frac{1}{2t}+c} = Ke^{-\frac{1}{2t}}, \qquad K := e^c \in \mathbb{R}_+ \\ y &= Ke^{-\frac{1}{2t}}, \qquad K := e^c \in \mathbb{R}. \end{aligned}$$

It must be $y(1) = Ke^{-\frac{1}{2}} \overset{!}{=} 1 \Rightarrow K = e^{\frac{1}{2}}$, so the solution is

$$y(t) = e^{\frac{1}{2} - \frac{1}{2}t}.$$

0.2. Determine the critical points of the following function and whether they are local maximum, minimum, or saddle points:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = y(x-1)e^{-(x^2+y^2)}.$$

Solution.

It is

$$Df(x,y) = \left(y\left(1 - 2x(x-1)\right) e^{-(x^2+y^2)}, (x-1)(1-2y^2)e^{-(x^2+y^2)} \right),$$

from which one sees that the critical points of f are

$$x_0 = (1,0), \qquad x_1 = \left(\frac{1+\sqrt{3}}{2}, \frac{\sqrt{2}}{2} \right), \qquad x_2 = \left(\frac{1+\sqrt{3}}{2}, -\frac{\sqrt{2}}{2} \right),$$
$$x_3 = \left(\frac{1-\sqrt{3}}{2}, \frac{\sqrt{2}}{2} \right), \quad x_4 = \left(\frac{1-\sqrt{3}}{2}, -\frac{\sqrt{2}}{2} \right).$$

We have that

$$\operatorname{Hess} f(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-1},$$

so, since the matrix has eigenvalues of opposite sign, $(1,0)$ is a saddle point. Similarly, the 2nd derivative test yields that x_2, x_3 are local minima and x_1, x_4 are local maxima.

0.3. For which $a \in \mathbb{R}$ there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ whose gradient is the vector field

$$(x,y,z) \mapsto \begin{pmatrix} \log(1+x^2) + ay^2 \\ xy + y^2 \\ z^3 \end{pmatrix} ? \tag{1}$$

For such a's, find one function f. How do any such two such solutions (for the same a) differ?

Solution. We need to understand whether there exists f so that

$$\frac{\partial f}{\partial x}(x,y,z) = \log(1+x^2) + ay^2,$$
$$\frac{\partial f}{\partial y}(x,y,z) = xy + y^2,$$
$$\frac{\partial f}{\partial z}(x,y,z) = z^3.$$

If f exists, it must be

$$\begin{aligned} f(x,y,z) &= \int_0^x \log(1+t^2)dt + axy^2 + h_1(y,z) \\ f(x,y,z) &= \frac{xy^2}{2} + \frac{y^3}{3} + h_2(x,z) \\ f(x,y,z) &= \frac{z^4}{4} + h_3(x,y). \end{aligned}$$

for some h_1, h_2, h_3. Evaluating in $z = 0$ gives

$$h_3(x,y) = \frac{xy^2}{2} + \frac{y^3}{3} + h_2(x,0) = \int_0^x \log(1+t^2)dt + axy^2 + h_1(y,0). \tag{2}$$

This implies that $axy^2 = \frac{xy^2}{2}$ and thus $a = \frac{1}{2}$ is the only possibility. Let us show that, for such a there exist f whose gradient is actually (1). From (2) follows that

$$h_2(x,0) = \int_0^x \log(1+t^2)dt$$

and consequently

$$f(x,y,z) = \frac{z^4}{4} + h_3(x,y) = \frac{z^4}{4} + \frac{xy^2}{2} + \frac{y^3}{3} + \int_0^x \log(1+t^2)dt + C$$

for some $C \in \mathbb{R}$. One checks immediately that f does indeed solve(1).

Finally, if f_1, f_2 are two solutions with the same gradient, it means that $f_1 - f_2$ has zero gradient and hence it is constant. So any two solutions differ from each other by a constant.

0.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \cos(x^2 + y^2)$$

Compute the Taylor polynomial of f of 3rd order at the origin.

Solution. The Taylor polynomial at 0 is given by

$$\begin{aligned} T_f(x,y) =& f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y \\ &+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial^2 x}(0,0)x^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial y \partial x}(0,0)xy + \frac{\partial^2 f}{\partial^2 y}(0,0)y^2\right) \\ &+ \frac{1}{6}\left(\frac{\partial^3 f}{\partial^3 x}(0,0)x^3 + \frac{\partial^3 f}{\partial y \partial^2 x}(0,0)yx^2 + \frac{\partial^3 f}{\partial^2 x \partial y}(0,0)x^2 y + \frac{\partial^3 f}{\partial y \partial x \partial y}(0,0)xy^2 \right. \\ &\left. + \frac{\partial^3 f}{\partial x \partial y \partial x}(0,0)x^2 y + \frac{\partial^3 f}{\partial^2 y \partial x}(0,0)xy^2 + \frac{\partial^3 f}{\partial x \partial^2 y}(0,0)xy^2 + \frac{\partial^3 f}{\partial^3 y}(0,0)y^3\right). \end{aligned}$$

We notice first that it is

$$\begin{aligned} f(0,0) &= 1 \\ \frac{\partial f}{\partial x}(0,0) &= -2x\sin(x^2+y^2)|_{(x,y)=(0,0)} = 0, \\ \frac{\partial f}{\partial y}(0,0) &= -2y\sin(x^2+y^2)|_{(x,y)=(0,0)} = 0, \end{aligned}$$

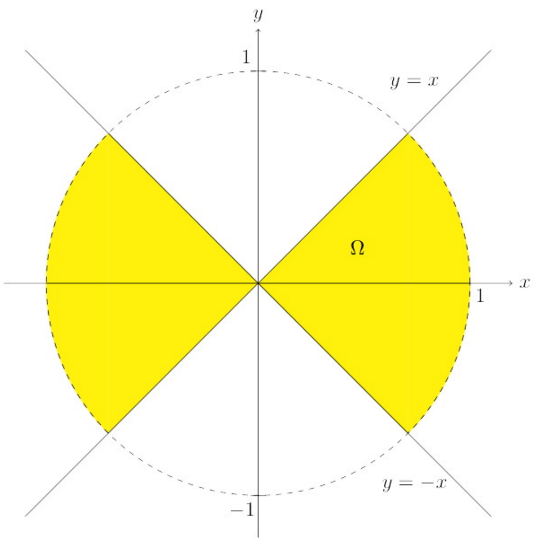
then we realize that, from the product rule, all 2nd and 3rd order derivatives vanish. Thus it is simply

$$T_f(x,y) = 1.$$

0.5. Integrate the function

$$f(x,y) = |x|\sqrt{x^2+y^2}$$

over the set Ω indicated below.



Solution. Ω consists of 4 parts:

$$\begin{aligned} \Omega_{++} &:= \{(x,y)\in\Omega \mid x,y\geq 0\}, \\ \Omega_{+-} &:= \{(x,y)\in\Omega \mid x\geq 0,\ y\leq 0\}, \\ \Omega_{--} &:= \{(x,y)\in\Omega \mid x,y\leq 0\}, \\ \Omega_{-+} &:= \{(x,y)\in\Omega \mid x\leq 0,\ y\geq 0\}, \end{aligned}$$

so we can divide the integral in

$$\int_\Omega f(x,y)\ dxdy = \int_{\Omega_{+-}} f(x,y)\ dxdy + \int_{\Omega_{++}} f(x,y)\ dxdy + \int_{\Omega_{--}} f(x,y)\ dxdy + \int_{\Omega_{-+}} f(x,y)\ dxdy.$$

Let $h(x,y) = (x,-y)$. Then there holds $h(\Omega_{++}) = \Omega_{+-}$ and the change of variable formula yields

$$\begin{aligned}\int_{\Omega_{+-}} f(x,y)\ dxdy &= \int_{h(\Omega_{++})} f(x,y)\ dxdy \\ &= \int_{\Omega_{++}} f\circ h(x,y)|\det(Dh)(x,y)|\ dxdy \\ &= \int_{\Omega_{++}} f(x,y)\ dxdy,\end{aligned}$$

where we used $f\circ h(x,y) = f(x,-y) = f(x,y)$ and $\det(Dh) \equiv -1$. Similarly one computes that

$$\int_{\Omega_{++}} f(x,y)\ dxdy = \int_{\Omega_{-+}} f(x,y)\ dxdy = \int_{\Omega_{--}} f(x,y)\ dxdy.$$

Whence

$$\int_\Omega f(x,y)\ dxdy = 4\int_{\Omega_{++}} f(x,y)\ dxdy.$$

On Ω_{++} we have $f(x,y) = x\sqrt{x^2+y^2}$. In polar coordinates the integral becomes

$$\int_{\Omega_{++}} f(x,y)dx\, dy \quad = \quad \int_0^1\int_0^{\frac{\pi}{4}} r^3\cos\phi dr\, d\phi = \left[\frac{r^4}{4}\right]_0^1 [\sin\phi]_0^{\frac{\pi}{4}} = \frac{1}{4\sqrt{2}},$$

and so

$$\int_\Omega f(x,y)\ dxdy = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

0.6. The Piriform curve is the planar curve given by

$$C := \{(x,y)\in\mathbb{R}^2 \mid y^2 = x^3(2-x)\}.$$

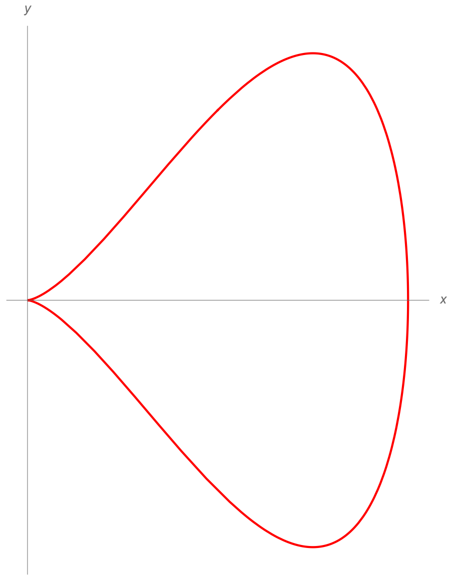


Figure 1: The Piriform Curve

A parametrization of C is given by $\gamma : [-\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}^2$,

$$\gamma(t) = \begin{pmatrix} 1+\sin(t) \\ \cos(t)(1+\sin(t)) \end{pmatrix}$$

Determine the area of the set Ω enclosed by C by means of Green's theorem.

Solution. First of all we notice that γ parametrises C in clockwise direction. Since C symmetric w.r.t. the x-axis, the curve $-\gamma$ is parametrised in the anti-clockwise direction. Since the areas enclosed by γ and $-\gamma$ are the same, using Green's theorem, we get:

$$\begin{aligned} \text{Area}(\Omega) &= -\int_{-\gamma} (y, 0) \cdot d\vec{s} = \int_{\gamma} (y, 0) \cdot d\vec{s} = \int_{-\pi/2}^{3\pi/2} \cos^2(t)(1+\sin(t))dt \\ &= \int_{-\pi/2}^{3\pi/2} \cos^2(t)\ dt + \int_{-\pi/2}^{3\pi/2} \cos^2(t)\sin(t)\ dt \\ &= \int_{-\pi/2}^{3\pi/2} \frac{1+\cos(2t)}{2}\ dt = \pi. \end{aligned}$$

0.7. Determine the values $\alpha, \beta \in \mathbb{R}$ for which integral in $\mathbb{R}^3$

$$\int_{\mathbb{R}^3} |x|^\alpha \mathrm{e}^{-|x|^\beta} d\mu$$

is convergent (no need to compute it).

Solution. The integral is positive, so it is convergent if and only if

$$\lim_{R\to\infty}\int_{B_R(0)}|x|^\alpha \mathrm{e}^{-|x|^\beta}d\mu$$

is finite. With a change of variables in spherical coordinates, we see that

$$\begin{aligned}\int_{B_R(0)}|x|^\alpha \mathrm{e}^{-|x|^\beta}d\mu &= \int_{B_R(0)} r^\alpha \mathrm{e}^{-r^\beta} r^2\sin\varphi dr d\vartheta d\varphi\\ &= C\int_0^R r^{\alpha+2}\mathrm{e}^{-r^\beta}\,dr\end{aligned}$$

where $C>0$ is a constant. So we are reduced to know for which values $\alpha,\beta\in\mathbb{R}$ The integral

$$\int_0^{+\infty} r^{\alpha+2}\mathrm{e}^{-r^\beta}\,dr$$

is convergent. We distinguish three cases.

If $\beta=0$, for no $\alpha\in\mathbb{R}$ the integral is convergent.

If $\beta>0$, with a change of variable $t=r^\beta$, the integral becomes

$$\frac{1}{\beta}\int_0^{+\infty} t^{\frac{3+\alpha}{\beta}-1}\mathrm{e}^{-t}\,dt,$$

which, from one variable calculus, is convergent if and only if $\frac{3+\alpha}{\beta}>0$, i.e. $\alpha>-3$.

If $\beta<0$, with a change of variable $t=r^\beta$ the integral becomes

$$-\frac{1}{\beta}\int_0^{+\infty} t^{\frac{3+\alpha}{\beta}-1}\mathrm{e}^{-t}\,dt,$$

which, from one variable calculus, is convergent if and only if $\frac{3+\alpha}{\beta}>0$, i.e. $\alpha<-3$.