Analysis II

Ordinary Diff equations.

ODE

Ex ① if $f'(t) - 2 = 0$

then what is f?

$f(t) = 2t + c$

∞'ly many solutions

$\left\{ \begin{array}{c} f'(t) - 2 = 0 \\ f(0) = 0 \end{array} \right\}$ Initial value problem.

$f(t) = 2t$ is the unique solution.

2) if $g'(t) - g(t) = 0$

what is g?

$g(t) = Ke^t$

∞'ly many solns.

$g(0) = 1. \Rightarrow K = 1$

$g(t) = e^t$ is the unique soln to the Initial value Prob.

$\left\{ \begin{array}{c} g'(t) - g(t) = 0 \\ g(0) = 1 \end{array} \right\}$.

Defn. In general an **ordinary** diff. equation (ODE) relates a function $f(x)$ at x, to the values of its derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$.

ie. It is an equation of the form

$$\circledast \quad F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

The function $x \mapsto f(x)$ is a solution of $\circledast$ in D if $F(x, f(x), \ldots, f^{(n)}(x)) = 0 \quad \forall x \in D$.

Defn. The order of the diff eqn is the highest order of derivative that appears in the equation.

Rk. A partial diff. eqn (PDE) is a diff. equation for a function of several variables. It involves "partial derivatives":

$$a \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

~~Ex. y'~~

Notation: Most of the time we just write f, $f', f'', \dots$ instead of $f(x), f'(x), \dots$

or $y, y', y'', \dots$ instead $y(x)$.

Ex: 1) $y' = 2xy$ ordinary diff. eqn of order 1

ODE order 3

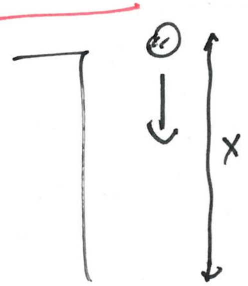
2) $y^{(3)} + 2xy'' + e^{x}y - 5 = 0$

3) $(\sin y)\, y' + (\tan x)\, y^2 + 1 = 0$.

ODE order 1

4) $f'(x+2) = f(x)$ is not an ord. Diff eqn.

Ex. Consider a mass falling under gravity.



Newton's law

$$F = ma$$

results in the eqn.

$$\cancel{m}\frac{d^2x}{dt^2} = -\cancel{m}g$$

↳ gravitational acc. $9.8\, m/sec^2$.

$$\Rightarrow \frac{dx^2}{dt^2} = -g$$

Integrating once $\frac{dx}{dt} = -gt + c$

Integrating a second time gives

$$x(t) = -g\frac{t^2}{2} + ct + D.$$

if $x(0) = x_0$

$\frac{dx}{dt}(0) = v_0$

then we can find c, D uniquely.

A very important example

① $y' = a(x)$

$a(x)$ is a fixed function which is continuous.

Fund. thm of Analysis I.

tells us that

$$y(x) = \int_{x_0}^{x} a(t)\,dt$$

is a solution.

② $y' = x + 1$

$$y(x) = \frac{x^2}{2} + x + c.$$

Linear differential equations.

Defn. A linear ODE of order k on I, is an eqn of the form

$$(*) \quad y^{(k)} + a_{k-1}(x)\, y^{(k-1)} + \cdots + a_1(x) y' + a_0 y = b(x).$$

where $b, a_1, \ldots a_{k-1}$ are continuous functions of x defined on I with values in $\mathbb{C}$

If $b=0$ then we say the ODE is homogenous otherwise inhomogeneous.

Rk. In a linear diff-eqn there no products of $y(x)$ with its derivatives, and all derivatives occur with power one.

$$(y'')^2 + yy' = 1$$

is not linear.

Also neither the function nor its derivatives are "inside" another function.

$\sin(y'') \cdot y' + e^{y} y = 0$.

linear

$\not\,$ ODE

Even simpler case is

the case of linear

ODE with constant

coefs.

$$y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_1 y' + a_0 y = b(x)$$

$a_1, \dots a_{k-1}$ are constants.

An initial condition

for the ODE ⊛.

is a set of equations

of the form

$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$y''(x_0) = y_2$$
$$\vdots$$
$$y^{k-1}(x_0) = y_{k-1}$$

<u>Rk</u> They are called linear

~~because~~ if we denote

LHS of ⊛ by $D(y)$

then

Then

$$D(\alpha f + \beta g)$$
$$= \alpha D(f) + \beta D(g).$$

$$D := \frac{d^{(k)}}{dx^k} + a_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \dots + a_0(x).$$

Main result about linear ODE's.

Main Thm 2.2.3 Let $I \subset \mathbb{R}$ open interval, $k \geq 1$ integer. Let

(*) $y^{(k)} + a_{k-1}(x) y^{k-1} + \dots + a_0 y = b$

be a linear ODE over I with continous coefficient. Then

*** ***

*** ***

① Let S_0 be the set of solutions when $b=0$ (i.e. the associated homog. ODE). Then S_0 is a vector space of dimension k. If $f_1 \dots f_k$ are the solns then so is $\alpha_1 f_1 + \dots + \alpha_k f_k$.

② For any initial conditions (i.e for any choice $x_0 \in I$ and $(y_0, \dots y_{k-1}) \in \mathbb{C}^k$ there is a unique solution $f \in S_0$ s.t.

$$f(x_0) = y_0$$
$$f'(x_0) = y_1$$
$$f^{(k-1)}(x_0) = y_{k-1}.$$

(3) For any arbitrary $b(x)$, the set of solutions of (*) is

$$S_b = \{ f + f_p \mid f \in S_0 \}$$

where f_p is a "particular" solution of (*)

(4) For any initial condition there is a unique solution $f \in S_b$. * *

Rk. Given a diff eqn like (*) we can always check whether a given function solves (*).

Ex. $y' + 3x^2 y = 0$.

We can check if e^{-x^3} is a solution or not

$$\left(e^{-x^3}\right)' = e^{-x^3} \cdot (-3x^2)$$

$$+ \quad 3x^2 y = 3x^2 e^{-x^3}$$

$$= 0. \checkmark$$

§ 2.3 Linear Diff. eqns of order 1.

$I \subset \mathbb{R}$ open interval

We'll consider the diff eqn of the form

(*) $y' + ay = b$

where $a(x)$, $b = b(x)$ are functions of x.

Rk. (*) $y^{(k)} + \cdots + a_0 y = b$

f_p is a soln of (*).

(1) $f_p^{(k)} + a_{k-1}(x) f_p^{k-1} + \cdots + a_0 f = b$

If f_0 is soln of

$y^{(k)} + \cdots + a_0 y = 0$.

then

(2) $f_0^{(k)} + \cdots + a_0 f_0 = 0$.

(1)+(2) $(f_p + f_0)^{(k)} + \cdots + a_0 (f_p + f_0)$

$= b + 0 = b$

$\Rightarrow$ $f_p + f_0$ also solves (*)

By Main Thm.

we should find

(1) Soln of the corresponding hom. eqn.

$y' + ay = 0$.

and

(2) Find a function f_p which satisfies

$$f_p' + a f_p = b.$$

Step 1 Homog soln.

$$y' + ay = 0.$$

$$y' = -ay$$

Assuming $y \neq 0$ on I

$$\frac{y'(x)}{y(x)} = -a(x)$$

$$\int \frac{y'(x)}{y(x)}\,dx = -\int a(x)\,dx.$$

$$\ln|y(x)| = -\int a(x)\,dx + C.$$

$$\int a(x)\,dx = A(x)$$

(ie $A'(x) = a(x)$).

$$\ln|y(x)| = -A(x) + C.$$

$$y(x) = e^{-A(x)} \cdot \underbrace{e^{C}}_{K}.$$

$$y(x) = K e^{-A(x)}$$

Ex: $y' + 2y = x$ $\quad a(x) = 2$, $b(x) = x$.

Associated hom. eqn.

$$y' + 2y = 0.$$

$$\frac{y'}{y} = -2$$

$$\ln|y| = -2x + C$$

$$y = Ke^{-2x}$$

Verify: $y' + 2y = 0$.

$$(Ke^{-2x})' + 2(Ke^{-2x}) = 0.$$

Prop. Any solution of

$$y' + a(x)y = 0 \text{ is}$$

of the form

$$f(x) = \underline{K}\, e^{-A(x)}$$

where $A(x) = \int a(x)\,dx$

If initial condition

$f(x_0) = y_0$ is given

then we can determine K.

The unique solution is then

$$f(x) = y_0 \exp(A(x_0) - A(x))$$

Step 2 Solution of inhom. eqn.

$$y' + a(x)y = b(x).$$

We need to find <u>one</u> particular function f_p such that

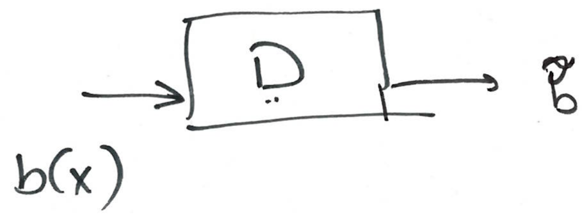
$$f_p' + a(x) f_p = b(x)$$

Then all the solutions are of the form

$$f = f_p(x) + f_0$$
$$= f_p(x) + K e^{-(A(x))},$$

To find f_p, there are 2 methods

(1) **Method 1** Educated guess.



Method is that we will assume that the soln will have a similar form to the function $b(x)$.

For example

if $b(x)$ is a poly of degree m

then we can try

$$f_p = c_m x^m + \cdots + c_0$$

if $b(x) = e^{3x}$ is an exponential function

we can try

$$f_p = e^{3x}$$

Ex: $y' + 2y = x$

$b(x) = x$

Method 1 for inhom. soln.

Assume f_p is also a poly of degree 1.

Assume $f_p = c_1 x + c_0$

$$(c_1 x + c_0)' + 2(c_1 x + c_0) = x$$

$$c_1 + 2c_1 x + 2c_0 = x.$$

$$2c_1 = 1 \qquad 2c_0 + c_1 = 0.$$

$$c_1 = \frac{1}{2} \qquad c_0 = \frac{-1}{4}$$

$f_p = \frac{1}{2}x - \frac{1}{4}$

Check! $\left(\frac{1}{2}x - \frac{1}{4}\right)' + 2\left(\frac{1}{2}x - \frac{1}{4}\right)$

$$\frac{1}{2} + x - \frac{1}{2} = x \checkmark$$

So any soln of

$$y' + 2y = x$$

is of the form

$$\boxed{\frac{1}{2}x - \frac{1}{4} + Ke^{-2x}}$$

Second method to solve inhom. eqn. is more systematic

Method 2. It is called "variation of parameters".

Idea: It assumes that a particular soln has a similar form to hom solution but the constant in the hom. soln will now be replaced by a function of x.

Let's assume f_p is of the form

$$f_p = K(x)\,e^{-A(x)}$$

We'll put this f_p into the diff eqn $\circledast$ and see what it forces $K(x)$ to satisfy.

Let's look at the example.

Ex $y' + 2y = x$ $\circledast$

$f_0 = K e^{-2x}$ is the hom. soln.

we guess for $f_p = K(x)e^{-2x}$.

Put this in $\circledast$

$$\left(K(x)e^{-2x}\right)' + 2\left(K(x)e^{-2x}\right) = x$$

$$K'(x)e^{-2x} + K(x)e^{-2x}(-2)$$
$$+ 2K(x)e^{-2x} = x.$$

$$K'(x)e^{-2x} = x$$

$$K'(x) = x e^{2x}$$

$$K(x) = \int x e^{2x}\,dx$$
$$\overset{\text{exercise}}{=} \left(\frac{1}{2}x - \frac{1}{4}\right)e^{+2x}.$$

$$f_p = K(x)e^{-2x} = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}\cdot e^{-2x}$$
$$= \frac{1}{2}x - \frac{1}{4}$$