ODE: • An equation for an unknown function f of the form

$$F(x, f(x), f'(x), \dots, f^{(n)}(x)) = 0$$

• f is a function of one variable
• The equation relates the values of f at a point, $f(x)$, to the values of its derivatives at the same point.

Order of the Diff. eqn = order of the highest derivative in the eqn.

Linear ODE

$$\boxed{y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = b} \quad (*)$$

where $y = f(x)$ is the function we want to find. $a_0(x), a_1(x), \dots, a_{k-1}(x), b(x)$ are functions.

$(*)$ is called **homogeneous** if $b \equiv 0$, **inhomogeneous** otherwise.

Main theorem for linear ODE's.

Let $I \subset \mathbb{R}$ be an open Interval, $k \geq 1$ integer

(*) $y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = b$, linear ODE

with continuous coefficients $a_{k-1}(x), \dots, a_0(x), b(x)$.

Let $S_0 := \{ f : I \to \mathbb{R} \mid f \text{ satisfies } (*) \text{ with } b \equiv 0 \}$

$\{ f : I \to \mathbb{R} \mid f^{(k)} + a_{k-1} f^{(k-1)} + \dots + a_0 f = 0 \}$.

Then ① S_0 is a vector space of dimension k = order of (*)

② For any initial conditions $y_0, \dots y_{k-1} \in \mathbb{C}^k$, there is a unique soln $f \in S_0$ such that

$f(x_0) = y_0, \quad f'(x_0) = y_1, \quad \dots, \quad f^{(k-1)}(x_0) = y_{k-1}$

③ For arbitrary $b(x)$ the set of solutions of (*) has the form $S_b = \{ f + f_p \mid f \in S_0 \}$

where f_p is one "particular" solution

④ For any initial condition, there is a unique soln $f \in S_b$

Rk S_b is NOT a vector space

Linear ODE of order 1

$y' + ay = b$ where $a = a(x)$, $b = b(x)$ are continuous functions.

Homogeneous eqn: $y' + ay = 0$

- Any solution is of the form $f(x) = K\exp(-A(x))$ where $A(x) = \int a(x)dx$ is a primitive of $a(x)$, $K \in \mathbb{C}$.
- The initial value problem $\left\{\begin{array}{l} y' + ay = 0 \\ y(x_0) = y_0 \end{array}\right\}$ has unique soln given by $f(x) = y_0 \exp(A(x_0) - A(x))$.

Inhomogeneous eqn $y' + ay = b$ (*)

- Assume $f_p(x) = z(x)\exp(-A(x))$ where $z(x)$ is a function
- Put $f_p(x)$ in (*) to see that $z'(x) = b(x)e^{A(x)}$
- Hence $z(x) = \int b(x)e^{A(x)}$, a primitive of $b(x)e^{A(x)}$

systematic method.

or educated guess. e.g. If $b(x)$ is a poly of degree n, we guess f_p is also

The same partikuler solution can also be obtained via "Integration factor"

(*) $\dfrac{dy}{dx} + a(x)\,y = b(x)$

We multiply both sides of the eqn (*) with

$$e^{A(x)} = e^{\int a(x)dx}$$

$$\frac{dy}{dx} e^{\int a(x)} + y\, a(x)\, e^{\int a(x)} = b(x) e^{\int a(x)}$$

Observe:

left hand side is

$$\frac{d}{dx}\left(y e^{\int a(x)} \right) = b(x) e^{\int a(x)}$$

Call $y e^{\int a(x)} := z(x)$ ☺.

$$\frac{d}{dx}\left(z(x)\right) = b(x) e^{\int a(x)}$$

then $z(x) = \int b(x) e^{A(x)}$

Now use ☺ $y = z(x) e^{-A(x)}$

$$y = \left(\int b(x) e^{A(x)}\right) e^{-A(x)}$$

Ex $\quad x\dfrac{dy}{dx} - 2y = x^2$

Assuming $x \neq 0$.

$$(*) \quad \boxed{\frac{dy}{dx} - \frac{2}{x}y = \frac{x^2}{x} = x}$$

$$\frac{dy}{dx} + a(x)y = b(x)$$

with $a(x) = -\dfrac{2}{x}$ $\quad b(x) = x$

First hom. soln.

$$\frac{dy}{dx} - \frac{2}{x}y = 0.$$

$$\frac{y'}{y} = \frac{2}{x}$$

$$\log|y| = \int \frac{2}{x} = 2\ln|x| + c$$
$$= \ln x^2 + c$$

$y = K\,x^2 = f_0$.

Check $\quad y' - \dfrac{2}{x}y = 0$

$$(Kx^2)' - \frac{2}{x}(Kx^2) = 0.$$

Particular soln: we try variation of parameters

Assume $f_p = K(x)\,x^2$

Put f_p into $(*)$.

$$\underbrace{K'(x)x^2 + K(x)\cdot 2x}_{y'} - \underbrace{\frac{2}{x}K(x)x^2}_{\frac{2}{x}y}$$

$$= \underbrace{x}_{b(x)}$$

$K'(x)x^2 = x$

$K'(x) = \frac{1}{x}$

$\Rightarrow K(x) = \ln|x|$

$f_p = x^2 \ln|x|$

general solution is then

$$f_p + f_0 = \underbrace{x^2 \ln|x|}_{f_p} + \underbrace{Kx^2}_{f_0}$$

Exercise : Solve it with integration factor

§2.4. Linear Diff eqns with constant coefs.

$$D = \frac{d^k}{dx^k} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + \cdots + a_0.$$

We're looking for solns of

$$Dy = b(x)$$

Here $a_0, a_1, \ldots a_{k-1}$ are constants.

Using main theorem

we need to find

(I) Soln of Homog. eqn

$D\cancel{f} = 0$ (H)

(II) Find a particular soln. f_p

Main thm $\Rightarrow$ general solution is of the form $f_p + f_0$

where $Df_0 = 0$.

Homogeneous eqn.

$$Df = 0 \qquad (H).$$

Recall: $y' + ay = 0$

$\Rightarrow y' = -ay$

$\Rightarrow y(x) = Ke^{-ax}$.

We say: what if the solution of

(H) $y^k + a_{k-1} y^{k-1} + \cdots + a_0 y = 0$

is also of the form

$y = e^{\lambda x}$ for some λ.

We put this "guess" back into (H)

$$(e^{\lambda x})^{(k)} + a_{k-1}(e^{\lambda x})^{(k-1)} + \dots + a_0 e^{\lambda x} = 0.$$

$$\lambda^k e^{\lambda x} + a_{k-1}\lambda^{k-1}e^{\lambda x} + \dots + a_0 e^{\lambda x} = 0.$$

$$e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + a_{k-2}\lambda^{k-2} + \dots + a_0) = 0$$

Since $e^{\lambda x} \neq 0$.

we must have

$$\lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_0 = 0$$

which means λ is a root of the polynomial,

$$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_0$$

<u>Defn</u>. For the linear ODE w/ constant coefficient

$$(H) \quad y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_0 y = 0$$

with $a_i \in \mathbb{C}$

The Polynomial

$$P_D(X) = X^k + a_{k-1} X^{k-1} + \dots + a_0$$

is called the companion or the characteristic polynomial of the hom. diff eqn (H)

The zeroes of $P(X)$ are called the eigenvalues of $D = \frac{d^k}{dx^{k-1}} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + \dots + a_0$.

Thm. $De^{\lambda x} = 0$

$\Leftrightarrow$ λ is a root of $P_D(X)$.

Fundamental theorem of algebra $\Rightarrow$

$P_D(X)$ has counting multiplicity has exactly k roots.

Examples

① $y'' - y = 0$

$$p(X) = X^2 - 1$$

$$= (X-1)(X+1)$$

$$\Downarrow$$

e^x, e^{-x} are solutions.

Any homog. soln. is a lin. comb. of these 2 solns

$$f_h(x) = c_1 e^x + c_2 e^{-x}.$$

② $y'' + y = 0$ $\quad y'' = -y$

$$p(X) = X^2 + 1$$

$$= (X-i)(X+i)$$

e^{ix}, e^{-ix}

$$f_h = c_1 e^{ix} + c_2 e^{-ix}$$

$$= c_1 [\cos x + i \sin x]$$

$$+ c_2 [\cos(-x) + i \sin(-x)].$$

$$= \underbrace{(c_1 + c_2)}_{d_1} \cos x + \underbrace{i(c_1 - c_2)}_{d_2} \sin x$$

$$= d_1 \cos x + d_2 \sin x$$

③ $y'' = 0$

$y(x) = c_1 + c_2 x$

$P(X) = X^2 = 0$

$e^{0x} = 1$, xe^{0x}

Thm. let $\alpha_1, \alpha_2 \ldots \alpha_r$ be pairwise distinct eigenvalues of

$$D : \frac{d^k}{dx^k} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + \cdots + a_0$$

(i.e: $\alpha_1 \ldots \alpha_r$ are roots of $X^k + a_{k-1}X^{k-1} + \cdots + a_0$)

with corresponding multiplicities $m_1, m_2, \ldots m_r$

Then the functions

$$f_{j,l} : \mathbb{R} \longrightarrow \mathbb{C}$$
$$x \longmapsto x^l e^{\alpha_j x}$$

$\cdot\ 1 \le j \le r$, $0 \le l < m_j$

form a system of fundamental solutions of $Df = 0$.

Rk) 1) If $\alpha \in \mathbb{R}$ is a root of $P(X)$ with multiplicity m, then corresponding to α we have solutions

$$e^{\alpha x}, \; x e^{\alpha x}, \; \dots \; x^{m-1} e^{\alpha x}$$

and they are linearly independent.

2) If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $\alpha = a + bi$ w/ multiplicity m, then the corresponding solutions

$$e^{\alpha x}, \; x e^{\alpha x}, \; \dots \; x^{m-1} e^{\alpha x}$$

they can also be written using

$$e^{\alpha x} = e^{(a+bi)x}$$

$$= e^{ax}\left[\cos bx + i \sin bx\right].$$

in terms of functions

$$e^{ax}\cos bx, \; e^{ax}\sin bx$$

$$x e^{ax}\cos bx, \; x e^{ax}\sin bx$$

$$\vdots$$

$$x^{m-1} e^{ax}\cos bx, \; x^{m-1} e^{ax}\sin bx$$

Clicker Question -

$$y''' - 2y'' + y' = 0.$$

$$P(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = 0.$$

$$\lambda(\lambda^2 - 2\lambda + 1)$$

$\lambda = 0 \quad m = 1 \rightarrow e^{0x} = 1$

$\lambda = 1 \quad m = 2 \rightarrow e^{x}$

xe^{x}.

$c_1 \cdot 1 + c_2 e^{x} + c_3 x e^{x}$ is the general homog. soln.

$\underline{Ex}$: $y^{(4)} + 2y'' - 8y' + 5y = 0$

$P(X) = X^4 + 2X^2 - 8X + 5$

General.

If you have a poly

$a_k x^k + \cdots + a_0 = 0.$

if $\frac{A}{B} \in \mathbb{Q}$, $\gcd(A, B) = 1$

is a soln

$a_k\left(\frac{A}{B}\right)^k + \cdots + a_0 = 0$

Then. $\underbrace{a_k A^k} + \underbrace{a_{k-1} A^{k-1} B} + \cdots + \underbrace{a_0 B^k} = 0$.

$A \mid a_k A^k$, $A \mid a_{k-1} A^{k-1} B$, $\cdots \Rightarrow A \mid a_0 B$

But $A \nmid B$

Hence $A \mid a_0$.

Similarly

$B \mid a_k$.

Possible rational roots of $X^4 + 2X^2 - 8X + 5$

are ± 1, ± 5

Can check easily

1 is a root

Ie $X-1$ divides $X^4 + 2X^2 - 8X + 5$

In fact $(X-1)^2$ divides

$P(X)$

$$P(X) = (X-1)^2 (X^2 + 2X + 5)$$

$\lambda = 1 \qquad m = 2$

$-1 \pm 2i$

$$y_H(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} \cos 2x + c_4 e^{-x} \sin 2x$$

If we also have the following conditions for the solution

$$\lim_{x \to \infty} y(x) = 0$$

$y(0) = 1$

$y'(0) = 3.$

$$\lim_{x \to \infty} y(x) = 0$$

$\Longrightarrow c_1 = 0, \quad c_2 = 0.$

$$y(0) = c_3 \cos 0 + c_4 \sin 0 = 1 = c_3$$

$y'(0) = \ldots$

find c_4.

Why $P_D(X)$ is called characteristic poly. and the roots eigenvalues like in Lin Alg?

~~Ex~~ Idea: A linear diff eqn of order k can be converted into a system of k linear D.E but each of which is now order 1.

Example

$$y'' + 3y' + y = 0.$$

$$P(X) = X^2 + 3X + 1$$

Let $y_1 := y$

$$\boxed{y_2 := y_1'} = y'$$

$$y_2' = y_1'' = y''$$

$$= -3y' - y.$$

$$\boxed{y_2' = -3y_2 - y_1}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Y' = AY.$$

$$P_A(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} -\lambda & 1 \\ -1 & -3-\lambda \end{pmatrix}$$

$$= \lambda(3+\lambda) + 1$$

$$= \underline{\lambda^2 + 3\lambda + 1}$$

<u>Part 2</u>. linear ODE w/ constant coef and <u>inhomogeneous</u>.

$$y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 = b(x)$$

<u>Goal</u>. To find f_p.

<u>Method I</u>: "<u>Method of undetermined coefficients</u>"

"an educated guess"

Idea: if the right hand side $b(x)$ is of special form then we search for a solution of similar form for the particular soln f_p

$b(x)$	"Ansatz" for f_p
$a e^{\alpha x}$	$C e^{\alpha x}$
$a \sin\beta x$ $a \cos\beta x$	$D \sin\beta x + E \cos\beta x$
$a e^{\alpha x} \sin\beta x$ $a e^{\alpha x} \cos\beta x$	$D e^{\alpha x} [\sin\beta x]$ $+ E e^{\alpha x} [\cos\beta x]$

$b(x)$	"Ansatz"
$P_n(x) e^{\alpha x}$ $\downarrow$ poly of degree n	$Q_n(x) e^{ax}$ $\downarrow$ poly of deg n.
$P_n(x) e^{\alpha x} \sin\beta x$ $P_n(x) e^{\alpha x} \cos\beta x$	$Q_n(x) e^{\alpha x} \sin\beta x$ $+ R_n(x) e^{\alpha x} \cos\beta x$.
$P_n(x) e^{\alpha x}$	$Q_n(x) e^{\alpha x}$,