Last time

6.10.22

Linear Diff. Eqns with constant coefficients

(I) Homogeneous eqn

$$y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = 0 \qquad (*)$$

Thm. Let $P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_0$ be the characteristic equation of $(*)$. Let $\alpha_1, \dots \alpha_r$ be pairwise distinct eigenvalues of $(*)$ with corresponding multiplicities $m_1, \dots, m_r$. Then the functions $f_{k,\ell} : \mathbb{R} \to \mathbb{C}$, $x \mapsto x^\ell e^{\alpha_j x}$

for $1 \le j \le r$, $0 \le \ell < m_j$ form a lin. independent system of solutions of $(*)$

Linear ODE with constant coefficients

$$y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = b(x) \quad (*)$$

$a_0, \dots a_{k-1}$ are constants

$b(x)$ is a continuous function.

To find a particular solution f_p we use "Method of undetermined coefficients":

① Choose an "Ansatz" which is similar to $b(x)$

② Put it into $(*)$, compare the coefficients of RHS and LHS to determine the consants in the Ansatz.

Rk

$(*)$

$b(x)$	Ansatz for f_p
$ae^{\alpha x}$	$ce^{\alpha x}$
$a\sin\beta x$ $a\cos\beta x$	$D\sin\beta x + E\cos\beta x$
$ae^{\alpha x}\sin\beta x$ $be^{\alpha x}\cos\beta x$	$e^{\alpha x}[D\sin\beta x + E\cos\beta x]$
$P_n(x)$	$R_n(x)$
$P_n(x)e^{\alpha x}$	$R_n(x)e^{\alpha x}$
$P_n(x)e^{\alpha x}\sin\beta x$ $Q_n(x)e^{\alpha x}\cos\beta x$	$e^{\alpha x}[R_n(x)\cos\beta x + S_n(x)\sin\beta x]$

Q_n, P_n, R_n, S_n polynomials of degree n.

Rk If $b(x)$ is a linear combination of these functions, use corresponding lin. comb. of Ansatz functions

Ex 1 $y'' + y' - 6y = 3e^{-4x}$

(H) $y'' + y' - 6y = 0$

$\Downarrow$

$\lambda^2 + \lambda - 6 = 0$

char. poly.

$(\lambda - 2)(\lambda + 3)$

$\lambda = 2, -3$

$y_H = c_1 e^{2x} + c_2 e^{-3x}$

For y_p particular solution

we try Ansatz $y_p = Ke^{-4x}$

$(Ke^{-4x})'' + (Ke^{-4x})' - 6Ke^{-4x}$

$= 3e^{-4x}$

$16Ke^{-4x} - 4Ke^{-4x} - 6Ke^{-4x}$

$= 3e^{-4x}$

$6Ke^{-4x} = 3e^{-4x}$

$\Rightarrow K = \frac{1}{2}.$

$y_p = \frac{1}{2} e^{-4x}$

General soln $y_H + y_p$,

$c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{2} e^{-4x}.$

2) $y'' + y' - 6y = 50\sin x$

$$y_H = c_1 e^{2x} + c_2 e^{-3x}$$

Ansatz:

$$y_p = K_1 \sin x + K_2 \cos x$$

$$y_p' = K_1 \cos x - K_2 \sin x$$

$$y_p'' = -K_1 \sin x - K_2 \cos x$$

$$-6y_p = -6K_1 \sin x - 6K_2 \cos x$$

+

$$50\sin x = (-7K_1 - K_2)\sin x + (K_1 - 7K_2)\cos x$$

$$\left.\begin{array}{l} -7K_1 - K_2 = 50 \\ K_1 - 7K_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} K_1 = -7 \\ K_2 = -1 \end{array}$$

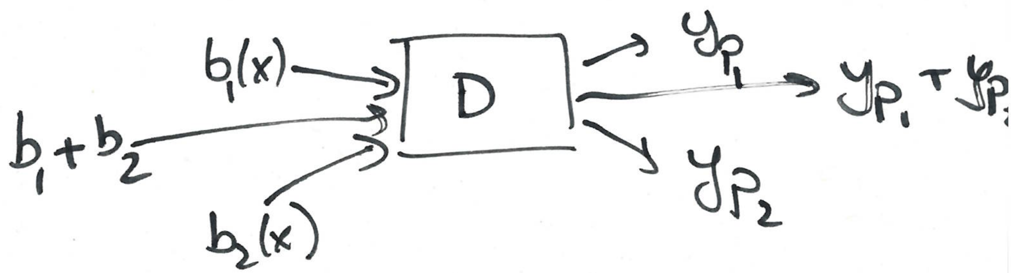
$$y_p = -7\sin x - \cos x$$

Rmk. If we had

$$y'' + y' - 6y = 50\sin x + 3e^{-4x}$$

$$y_p = \frac{1}{2} e^{-4x} + (-7\sin x - \cos x)$$

("Superposition principle).



Exercise Try $y_p = K\sin x$

3) $y'' + y' - 6y = 10e^{2x}$

Recall $y_H = c_1 e^{2x} + c_2 e^{-3x}$.

If we try $y_p = Ke^{2x}$

$$y_p'' + y_p' - 6y_p = 4Ke^{2x} + 2Ke^{2x} - 6Ke^{2x}$$
$$= 0 \; !$$

Try instead

$$y_p = Kxe^{2x}$$

$$y_p' = Ke^{2x} + 2Kxe^{2x}$$
$$y_p'' = 2Ke^{2x} + 2K\left[e^{2x} + 2xe^{2x}\right]$$
$$-6y_p = -6Kxe^{2x}$$

$$5Ke^{2x} + [2K + 4K - 6K]xe^{2x}$$
$$= 10e^{2x}$$

$$\Rightarrow K = 2.$$

$$y_p = 2xe^{2x}.$$

general soln

$$y = y_H + y_p$$
$$c_1 e^{2x} + c_2 e^{-3x} + 2xe^{2x}$$

In general if $b(x)$ is a solution of homog. equation. Try $x\,b(x)$, $x^2 b(x)$, .. etc, depending on the multiplicity of the eigenvalue.

Rmk In the "Ansatz" method of the disturbance function is of the form

$e^{\alpha x}$ $\alpha \in \mathbb{C}$

for which α is also a zero of the charac. polynomial ie $e^{\alpha x}$ is a solution of the homog. equation

then if α is a root of multiplicity one then as "Ansatz" we try

$x e^{\alpha x}$

If α is a root of multiplicity 2, then $e^{\alpha x}$, $x e^{\alpha x}$ are both solns of homog eqn, so try the next one $Kx^2 e^{\alpha x}$

Clicker $\quad y'' - 4y = 4e^{2x}$

$$\cancel{x}^2 - 4 = 0$$

$$(\lambda - 2)(\lambda + 2)$$

$$y_H = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = Kxe^{2x}$$

$$y_p' = Ke^{2x} + 2Kxe^{2x}$$

$$y_p'' = 2Ke^{2x} + 2K[e^{2x} + 2xe^{2x}].$$

$$-4y_p = -4Kxe^{2x}$$

$$4e^{2x} = 4Ke^{2x}$$

$K = 1.$

$$y_p = xe^{2x}$$

$$y_{general} = c_1 e^{2x} + c_2 e^{-2x} + xe^{2x}.$$

$$xe^{2x} + e^{2x}$$

$$xe^{2x} + e^{-2x}$$

$$xe^{2x} + \cancel{2}e^{2x} + \pi e^{-2x}$$

are all solutions

Separation of Variables.

A diff. eqn of first order is called separable if it is of the form

$$y' = b(x)\,g(y)$$

We can "separate" the variables to write

$$\frac{dy}{dx} = b(x)\,g(y)$$

$$\frac{dy}{g(y)} = b(x)\,dx$$

As long as $g(y) \neq 0$.

If $g(y_0) = 0$ then $y = y_0$ is a soln.

Ex: $e^{2y}\,y' = x$

with $x > 0$.

$$\frac{dy}{e^{-2y}} = x\,dx$$

Integrate both sides

$$\int \frac{dy}{e^{-2y}} = \int x\,dx$$

$$\frac{e^{2y}}{2} = \frac{x^2}{2} + c$$

$$e^{2y} = x^2 + \tilde{c}$$

$$2y = \log(x^2 + c)$$

$$y = \frac{1}{2}\log(x^2+c)$$

(2) $y' = 1 + y^2$ $(1+y^2)$ is never zero.

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{dy}{1+y^2} = \int dx \Rightarrow \arctan y = x + c$$

$$y = \tan(x+c).$$

Differential calculus in $\mathbb{R}^n$.

In Analysis I we've seen functions of 1 variable

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x)$$

$$f : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$x \longmapsto f(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

$$f_i : \mathbb{R} \longrightarrow \mathbb{R}.$$

f is continous $\Leftrightarrow$ f_i is continous $\forall i$

f is diff $\Leftrightarrow$ f_i is diff $\forall i$

We want to study of functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

when $n > 1$

if $m > 1$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longmapsto (f_1(x), f_2(x), \dots f_m(x))$$

with $f_i(x) : \mathbb{R}^n \longrightarrow \mathbb{R}.$

For $m > 1$ a function

$f : \mathbb{R}^n \to \mathbb{R}^m$ is

also called a vector field.

$f : \mathbb{R}^n \to \mathbb{R}$ is

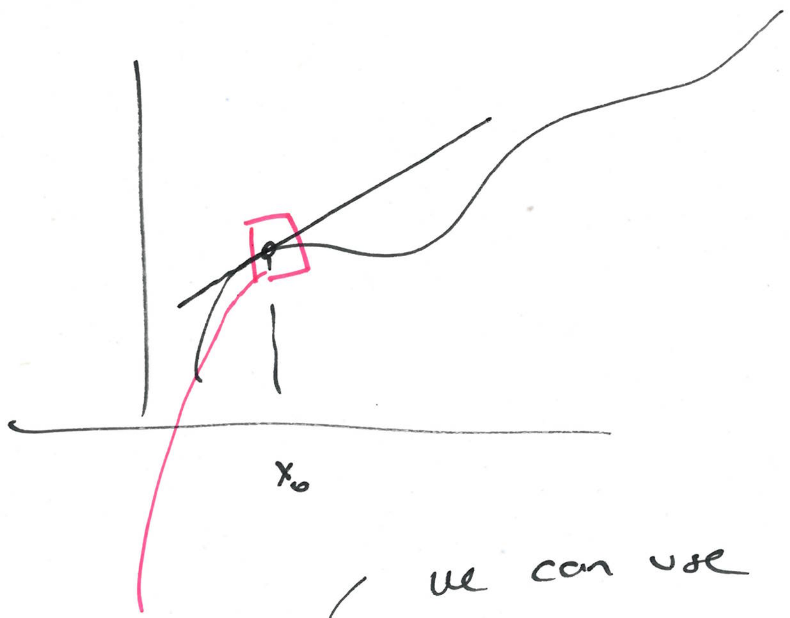
sometimes called a scalar field.

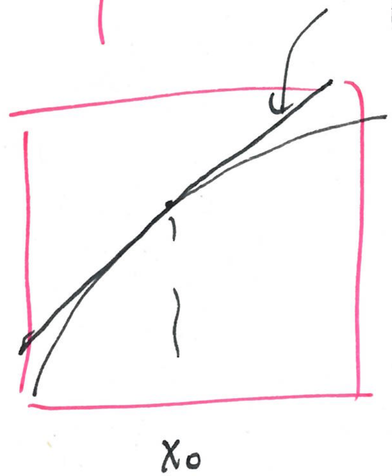
Examples ① Linear maps.

$$f : \mathbb{R}^n \to \mathbb{R}^m$$
$$x \mapsto Ax$$

for a $m \times n$ matrix A.

Recall





we can use the tangent line as an approx. to the function near x_0.

A bit more generally

Affine maps

$f: \mathbb{R}^n \to \mathbb{R}^m$

$x \to Ax + y_0$

where $y_0 \in \mathbb{R}^m$ is a fixed vector in $\mathbb{R}^m$.

(2) Quadratic forms.

$Q: \mathbb{R}^n \to \mathbb{R}$.

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

$n=2$

$$Q(x,y) = ax^2 + bxy + cy^2$$

diagonal quadratic form

$$Q(x_1, x_2, \ldots x_n) = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Polynomial in n variables.

Given $d \geq 0$ a polynomial in n variables of degree $\leq d$ is a finite sum of monomials of degree $\leq d$.

A monomial is a function of the form

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, x_2, \dots, x_n) \longmapsto \alpha x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

degree of $(x_1^{d_1} \cdots x_n^{d_n}) = d_1 + d_2 + \dots + d_n$

$$P(x, y, z) = x^3 + 2x^2 y + xyz + 5z^4$$

degree of $P = 4$

In general degree of a Poly. of several variables is the degree of its highest degree monomial.

We can define new functions from old ones in various ways

① Cartesian Product of 2 functions

$$f_1 : \mathbb{R}^n \to \mathbb{R}^s, \quad x \to f_1(x)$$
$$f_2 : \mathbb{R}^n \to \mathbb{R}^t, \quad x \to f_2(x)$$
$$f : \mathbb{R}^n \to \mathbb{R}^{s+t}, \quad x \longmapsto (f_1(x), f_2(x))$$

Any function $f : \mathbb{R}^n \to \mathbb{R}^m$
$x \to (f_1(x), \ldots, f_m(x))$

is in fact a

cartesian product

of functions $f_1, \ldots f_m$

$f_i : \mathbb{R}^n \to \mathbb{R}$.

② Composition of functions.

$f : \mathbb{R}^n \to \mathbb{R}^m \qquad g : \mathbb{R}^m \to \mathbb{R}^t$

$g \circ f : \mathbb{R}^n \to \mathbb{R}^t$

ex $g : \mathbb{R}^n \to \mathbb{R}$
$(x_1, \ldots x_n) \mapsto x_1^2 + x_2^2 + \cdots + x_n^2$

$f : \mathbb{R} \to \mathbb{R}$
$x \to e^x$

$(f \circ g)(x_1, \ldots x_n)$
$= e^{x_1^2 + \cdots + x_n^2}$.