New Functions from old.

1) Cartesian product of functions

$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}.$$
$$x \longmapsto f_i(x)$$

$$f = (f_1, f_2, \dots f_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longmapsto (f_1(x), \dots, f_m(x))$$

2) composition of 2 functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad , \quad g : \mathbb{R}^m \longrightarrow \mathbb{R}^s$$

$$(g \circ f) : \mathbb{R}^n \longrightarrow \mathbb{R}^s$$
$$x \longmapsto g(f(x)) .$$

3) if $f_1, f_2, \dots, f_n$ are functions on $\mathbb{R}$ we can define

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x = (x_1, \dots x_n) \longmapsto f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

f is called function with seperated variables

e.g. Monomials.

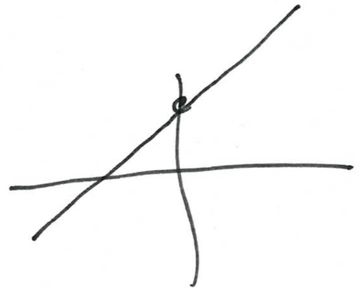
$$M : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x_1, \dots x_n) \longmapsto x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

Ways of visualizing functions

① Graph of f

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}.$

$\operatorname{graph}(f) := \{(x, y, f(x,y)) \in \mathbb{R}^3\}.$



② . Parametric plot.

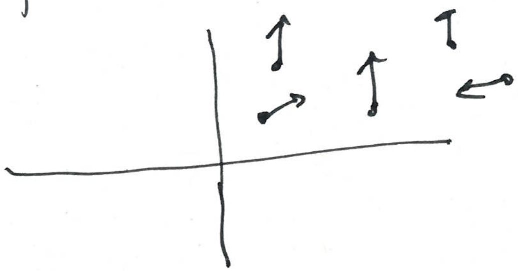
$f : \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (f_1(t), f_2(t))$

$f : \mathbb{R} \longrightarrow \mathbb{R}^3$

eg : $t \longrightarrow \underbrace{(t, t+1)}_{y = x+1}$

③ Vector Plot.

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

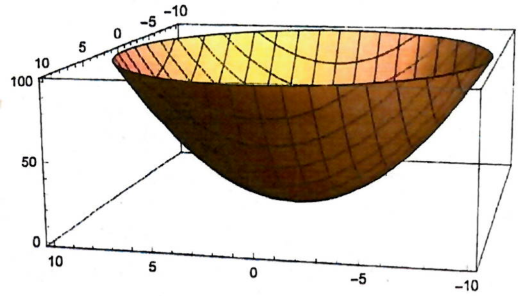


See the pictures.

Graph of Functions $f: \mathbb{R}^2 \to \mathbb{R}$.

```
In[9]:= Plot3D[x^2 + y^2, {x, -10, 10}, {y, -10, 10},
          RegionFunction → Function[{x, y, z}, x^2 + y^2 < 100]]
```

Out[9]=

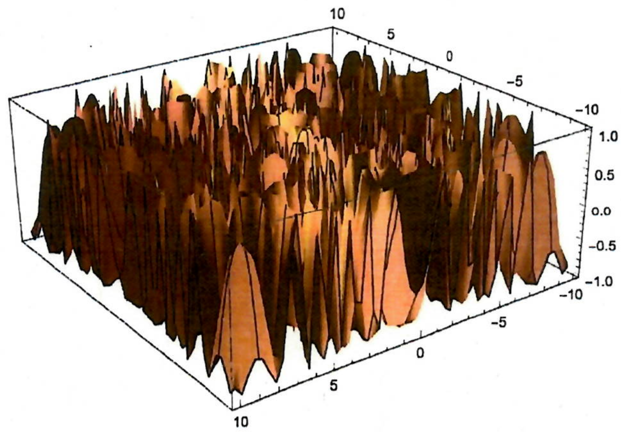


$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \Rightarrow x^2+y^2$

```
In[10]:= Plot3D[Sin[x^2 + y^2], {x, -10, 10}, {y, -10, 10}]
```

$f: \mathbb{R}^2 \to \mathbb{R}$
$x,y \to \sin(x^2+y^2)$.

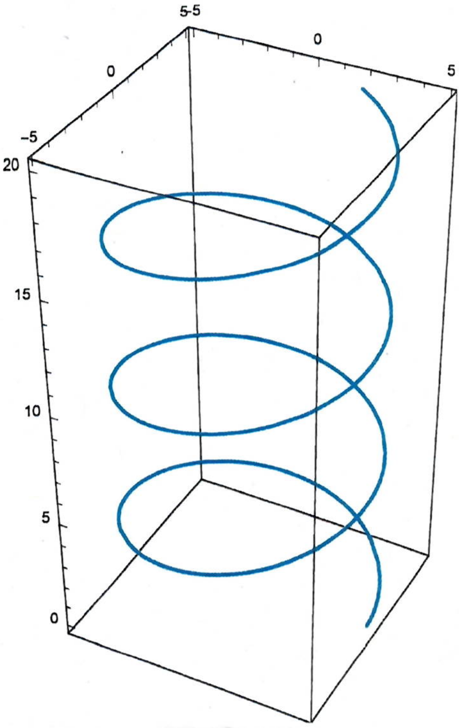
Out[10]=



2.1

In[27]:= `ParametricPlot3D[{5 Cos[t], 5 Sin[t], t}, {t, 0, 20 }]`

Out[27]=

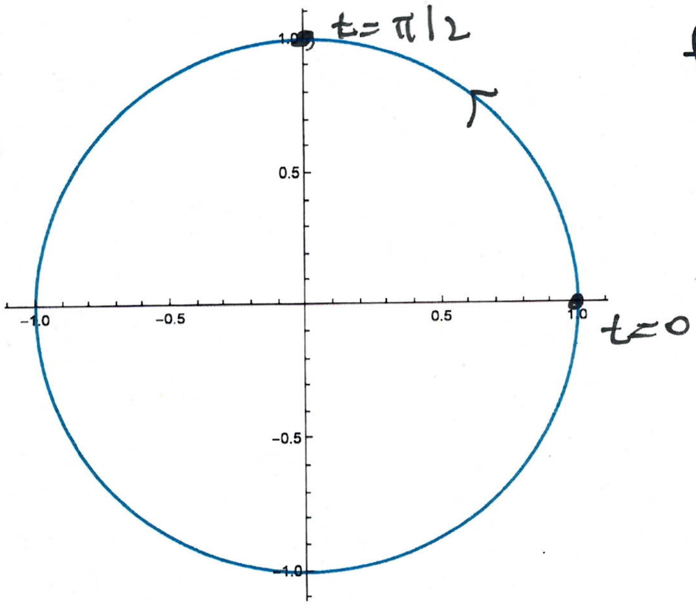


$$g(t) = (5\cos t, 5\sin t, t)$$

$$g : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (5\cos t, 5\sin t, t)$$

In[21]:= `ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 10}]`

Out[21]=

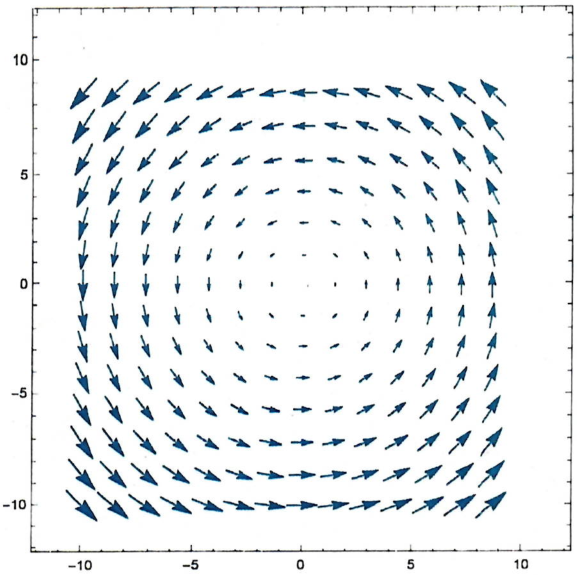


$$f(t) = (\cos t, \sin t)$$

$$f : \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

```
In[5]:= VectorPlot[{-y, x}, {x, -10, 10}, {y, -10 , 10 }]
```

Out[5]=

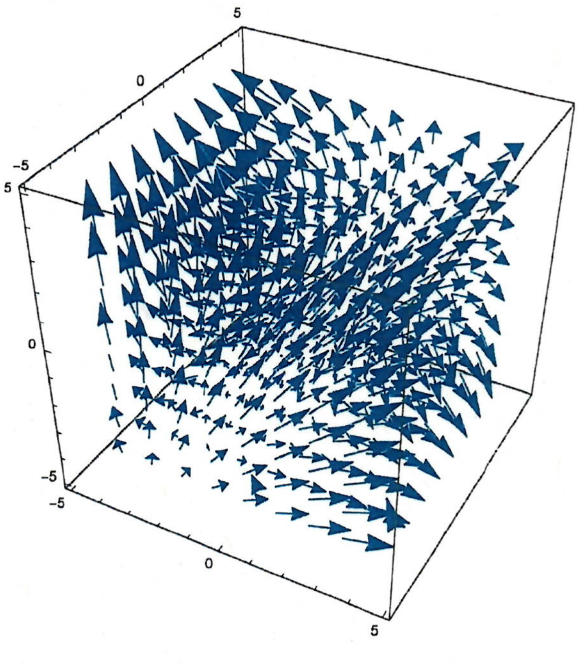


$f = \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to (-y, x)$

```
In[2]:= VectorPlot3D[{x, -y, z + 1}, {x, -4, 4}, {y, -4, 4}, {z, -4, 4}]
```

Out[2]=

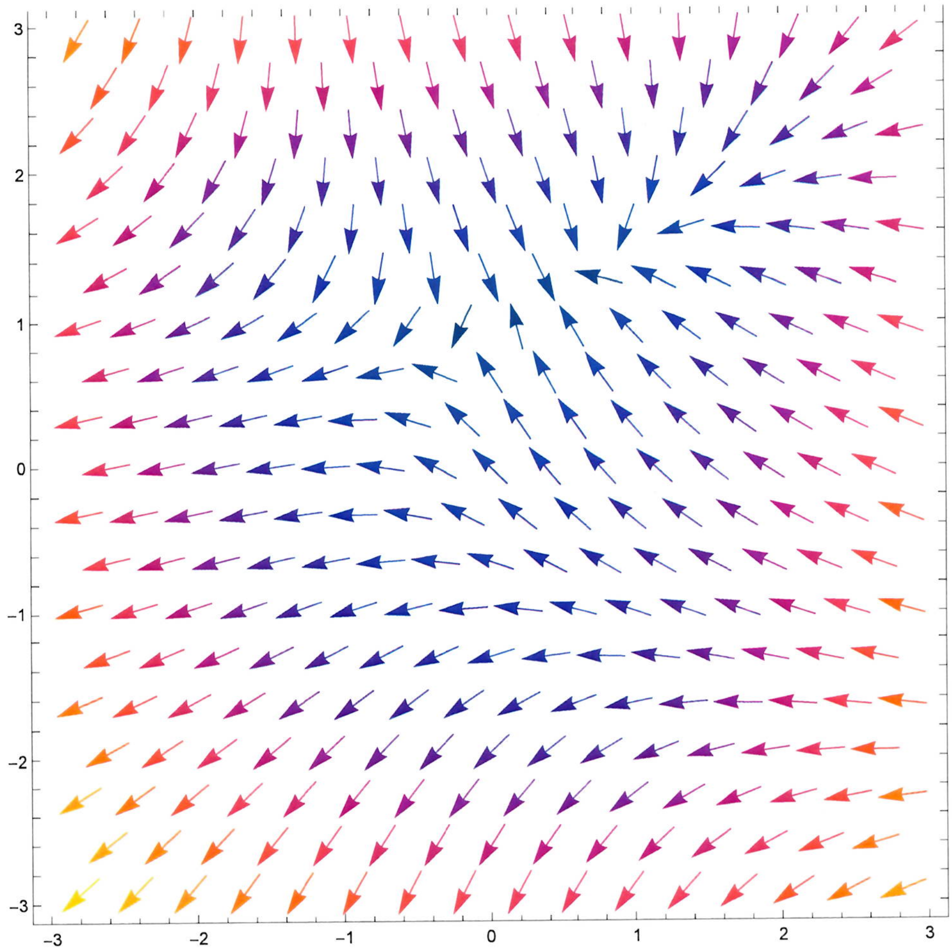


$f = \mathbb{R}^3 \to \mathbb{R}^3$

$(x, y, z) \to (x, -y, z+1).$

In[2]:= `VectorPlot[{-x^2 + y - 1, x - y^2 + 1}, {x, -3, 3}, {y, -3, 3}]`

Out[2]=



$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \mapsto (-x^2 + y - 1, \; x - y^2 + 1).$$

§3.2 Continuity in $\mathbb{R}^n$.

Recall Any function $f: \mathbb{R}^n \to \mathbb{R}^m$

$$x \mapsto (f_1(x), \dots f_m(x))$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$.

Recall $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

f cont. means : $\boxed{\lim_{x \to x_0} f(x) \text{ exists } \underline{\text{and}} \text{ equal to } f(x_0).}$

We can also look at the defn using sequences:

For every sequence $(x_n)_n$ with limit x_0 we have that the sequence $f(x_n)$ converges to $f(x_0)$

$$(x_n) \to x_0 \Rightarrow f(x_n) \to f(x_0)$$

$$\boxed{\lim f(x_n) = f(\lim x_n)}$$

To define limit and continuity for $f : \mathbb{R}^n \to \mathbb{R}^m$

we recall:

For $x \in \mathbb{R}^n$, $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$

$\|x\| \geq 0 \quad \forall x \neq 0$

$\|x\| = 0 \Leftrightarrow x = 0$.

$\|t x\| = |t| \, \|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

<u>Defn</u> let $(x_k)_k \subset \mathbb{R}^n$ be a sequence in $\mathbb{R}^n$

$x_k = (x_{k,1}, x_{k,2}, \ldots x_{k,n})$

let $y = (y_1, \ldots y_n) \in \mathbb{R}^n$

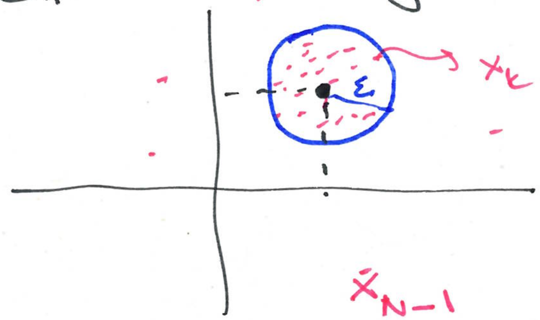
we say the sequence (x_k) converges to y as $k \to \infty$, we write

$\boxed{x_k \to y}$ if

$\forall \varepsilon > 0 \quad \exists N \geq 1$ s.t

$\forall k \geq N \quad \|x_k - y\| < \varepsilon$

$(x_k) \in \mathbb{R}^2$, $y = (1,1)$



Lemma Let (x_k) be a sequence in $\mathbb{R}^n$ $(y_1, \dots, y_n) = y \in \mathbb{R}^n$. Then FAE (following are equivalent).

① $\lim x_k = y$

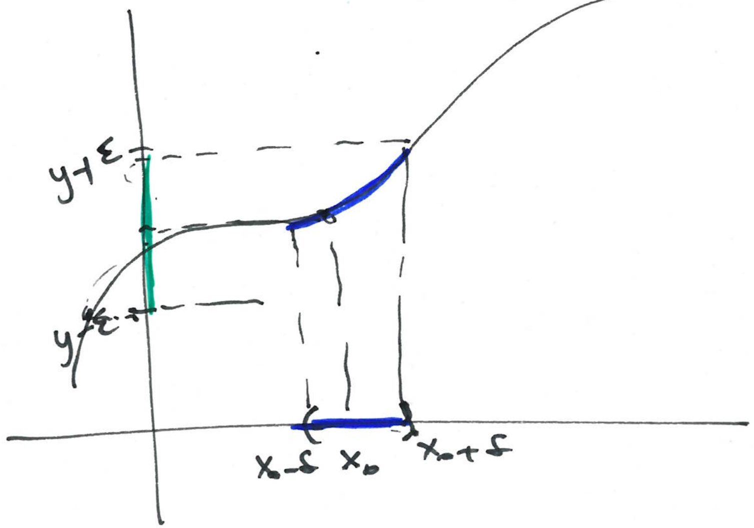
② For each i, $1 \leq i \leq n$ the sequence $(x_{k,i})_k \subset \mathbb{R}$ of real numbers converge to $y_i \in \mathbb{R}$.

③ The sequence of real numbers $\|x_k - y\| \to 0$.

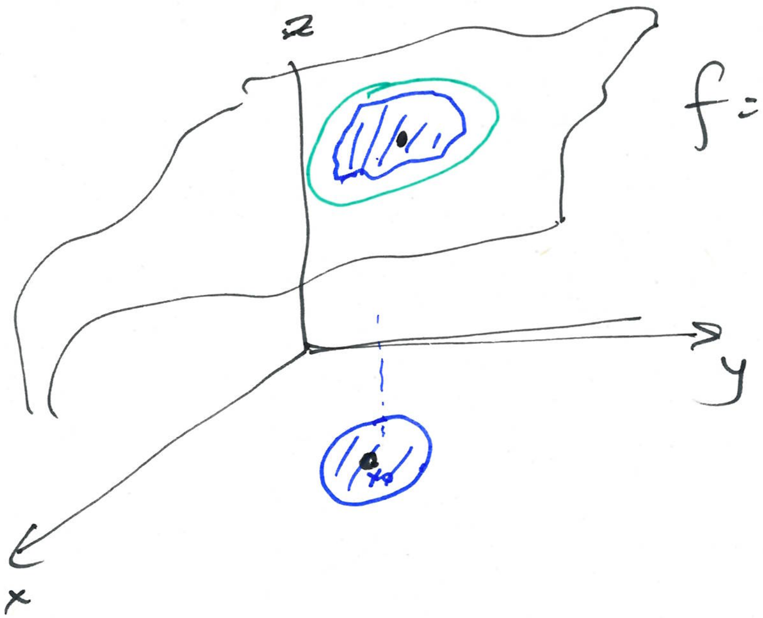
Defn. Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ $x_0 \in X$, $y \in \mathbb{R}^m$

We say f has a limit as $x \to x_0$ (with $x \neq x_0$) y if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in X$, $x \neq x_0$ such that $\|x - x_0\| < \delta$ we have $\|f(x) - y\| < \varepsilon$

We write $\boxed{\lim_{x \to x_0} f(x) = y.}$



$f: \mathbb{R} \to \mathbb{R}$



$f: \mathbb{R}^2 \to \mathbb{R}$

Prop. $f: X \to \mathbb{R}^m$

$X \subset \mathbb{R}^n$

$x_0 \in X$, $y \in \mathbb{R}^m$

$$\lim_{x \to x_0} f(x) = y$$

$\Longleftrightarrow$ $\forall$ sequence (x_k) with

$\lim x_k = x_0$

we have

$\lim f(x_k) = y$.

Defn let $x_0 \in X$

$f : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$

f is continuous at x_0

if: $(\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon)$

f is continuous on X

if it is continuous $\forall x_0 \in X$

Prop f is cont at x_0

$\Longleftrightarrow$ $\forall$ sequence $(x_k) \subset X$

with $\lim x_k = x_0$ we have

$\lim f(x_k) = f(x_0)$

ie $\boxed{\lim f(x_k) = f(\lim x_k)}$.

Examples of continuous functions

① Cartesian product of continuous funcs are continuous.

if $f_1 : \mathbb{R}^n \to \mathbb{R}^m$

$f_2 : \mathbb{R}^n \to \mathbb{R}^s$

continuous

then $f = (f_1, f_2) : \mathbb{R}^n \to \mathbb{R}^{m+s}$

$x \mapsto (f_1(x), f_2(x))$

is continuous.

(2) A function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$
$$x \to (f_1(x), f_2(x), \dots f_m(x))$$

Is continous

$$\iff f_i : \mathbb{R}^n \to \mathbb{R}$$
$$x \mapsto f_i(x)$$

are continuous $\forall$

$1 \leq i \leq m$.

(3) Any linear map

$$f : \mathbb{R}^n \to \mathbb{R}^m$$
$$x \mapsto Ax$$

for some $A_{m \times n}$

Is continuous.

It is enough to consider the case $m = 1$

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1 \dots x_n) \mapsto a_1 x_1 + \dots + a_n x_n.$$

If (x_k) is a sequence conv. to $y = (y_1, \dots y_n)$

Then $(x_k)_k = \left((x_{k,1}, \dots x_{k,n})\right)_k$

$$\lim_{k\to\infty} x_{k,i} = y_i \qquad 1 \leq i \leq n.$$

and $\lim a_i x_{k,i} = a_i y_i$

$$f(x_k) = a_1 x_{k,1} + a_2 x_{k,2} + \dots + a_n x_{k,n}$$

$$\to a_1 y_1 + \dots + a_n y_n$$
$$= f(y)$$

We are using

$a_i \to a$ then $a_i + b_i \to a+b$
$b_i \to b$

4 Sums and products of finitely many cont. functions are continuous.

5 Functions w/ separated variables are continuous if each factor is continuous.

$$f(x_1, \dots x_n) = f_1(x_1) \dots f_n(x_n)$$

In particular Polynomials are continuous.

(6) Composition of continuous functions are continuous.

let $(x_n) \to x$

f cont : $f(x_n) \to f(x)$

g cont

$f : \mathbb{R}^n \to \mathbb{R}^m$

$g : \mathbb{R}^m \to \mathbb{R}^t$

w.t.s $(g \circ f)$ is continuous at x

w.t.s. $(g \circ f)(x_n) \to (g \circ f)(x)$.

Since f is cont.

we know: $f(x_n) \to f(x)$

g cont $\Rightarrow g(f(x_n)) \to g(f(x))$.

i.e. $(g \circ f)(x_n) \to (g \circ f)(x)$

(7) If $f : \mathbb{R}^2 \to \mathbb{R}$

is continuous. Fix $y_0 \in \mathbb{R}$

Define $g_{y_0}(x) := f(x, y_0)$

Then $g_{y_0} : \mathbb{R} \to \mathbb{R}$ is continuous.

Since it is composition of f with the function $x \longmapsto (x, y_0)$

Warning! Converse is not true!

~~Assume~~ Assume: for a given $f: \mathbb{R}^2 \to \mathbb{R}$ the function $g_{y_0}: \mathbb{R} \to \mathbb{R}$, $x \to f(x, y_0)$ is continous $\forall\, y_0 \in \mathbb{R}$

Can we conclude that $f: \mathbb{R}^2 \to \mathbb{R}$ is also continuous?

NO!

eg **Defn:** $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} x & \text{if } y \geq 0 \\ -x & \text{if } y < 0 \end{cases}$$

For each $y_0 \geq 0$

$$g_{y_0}: \mathbb{R} \to \mathbb{R}$$
$$x \to f(x, y_0) = x$$
cont.

if $y_0 < 0$

$$g_{y_0} : \mathbb{R} \to \mathbb{R}$$
$$x \to -x$$

is also cont.

(See the picture)

But f is not continuous.

For example $f(1,0) = 1$

Take the sequence

$$\left(\left(1, -\frac{1}{k}\right)\right)_k \to (1, 0)$$

$$f\left(\left(1, -\frac{1}{k}\right)\right) = -1$$

$$\lim f\left(\left(1, -\frac{1}{k}\right)\right) = -1 \neq 1 = f\left(\lim\left(1, -\frac{1}{k}\right)\right)$$
$$= f(1,0)$$

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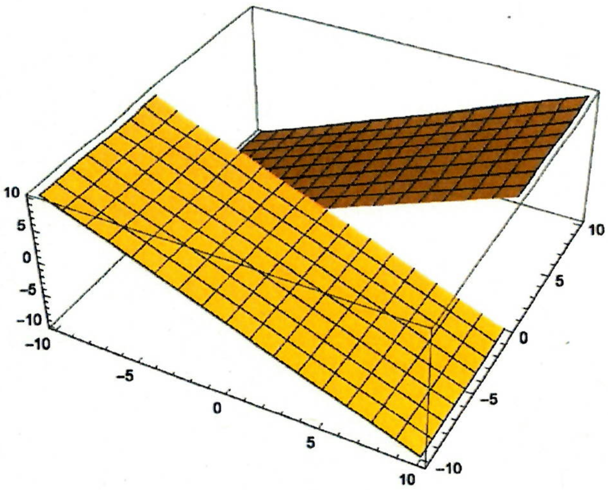
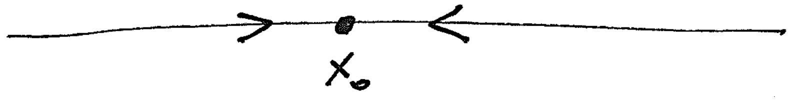
```
In[7]:= f[x_, y_] := Piecewise[{{-x, y < 0}, {x, y > 0}}]
        Plot3D[f[x, y], {x, -10, 10}, {y, -10, 10}]
```

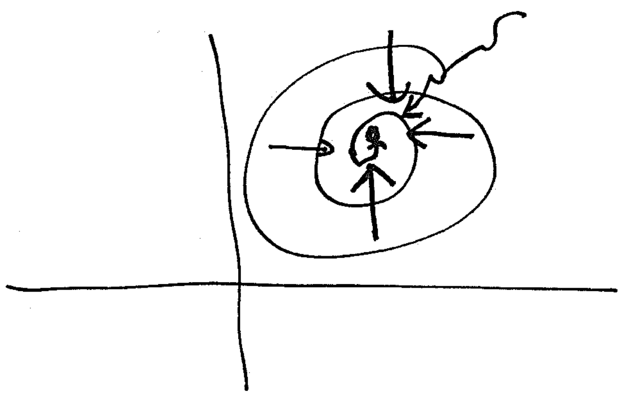
Out[8]=



$f(x,y)$ is not continous as a function of 2 variables

even though $g(x) = g_{y_0}(x) := f(x, y_0)$ for any y_0 is continous.



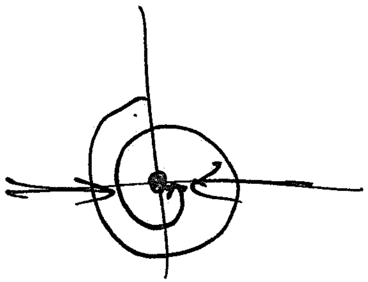


<u>Rk.</u> For functions of 1 variable, when we are checking if the limit exists at one point, we really only need to check as we approach from right or from left. These are the only possible ways we can approach a point on the real line

But when the point is in $\mathbb{R}^n$, $n > 1$ there are ∞ly many ways we can approach the pt.

<u>eg.</u> We can approach $(0,0)$ via $(k,0)$, $k \to 0$

via $(0,k)$



or $\frac{\cos k}{k}, \frac{\sin k}{k}$

$k \to \infty$.

<u>Ex.</u>

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

If we approach along x-axis

$$\lim_{(x,0) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x \cdot 0}{x^2+0} = 0.$$

$$\lim_{(0,y) \to (0,0)} f(x,y) = \lim_{y \to 0} \frac{0 \cdot y}{0^2+y^2} = 0.$$

If we approach along the line

$$y = mx$$

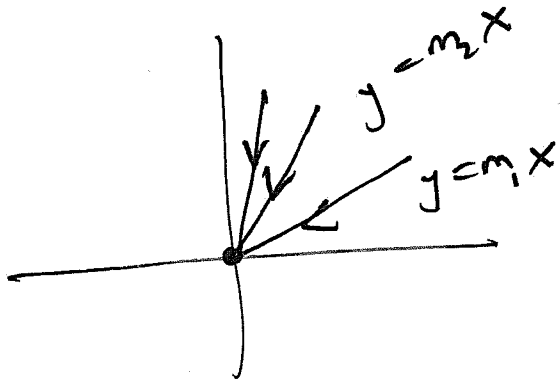
$$\lim_{(x,mx)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

changes as we change the slope.



Hence f does not have a limit at $(0,0)$

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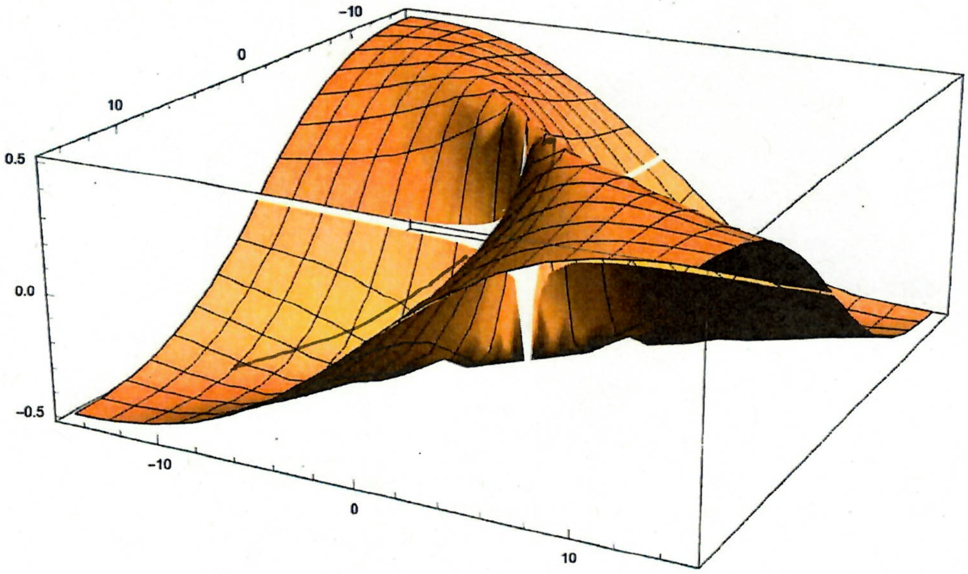
```
In[10]:= Plot3D[x * y / (x^2 + y^2), {x, -15, 15}, {y, -15, 15}, Exclusions → {x == 0, y == 0}]
```

Out[10]=



$f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Looks like it is discontinuous at (0,0)

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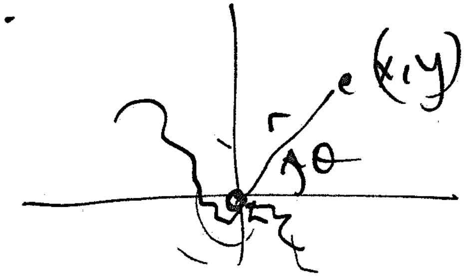
For functions of 2 variables sometimes polar coordinates is helpful.



$x = r\cos\theta \qquad y = r\sin\theta$

$$\lim_{(r\cos\theta,\, r\sin\theta)\to(0,0)} f(x,y) = \lim \frac{(r\cos\theta)(r\sin\theta)}{r^2}$$

$$= \lim_{r\to 0} \cos\theta \sin\theta$$

This has different values for different θ's.

Ex. $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

If we use polar coordinates

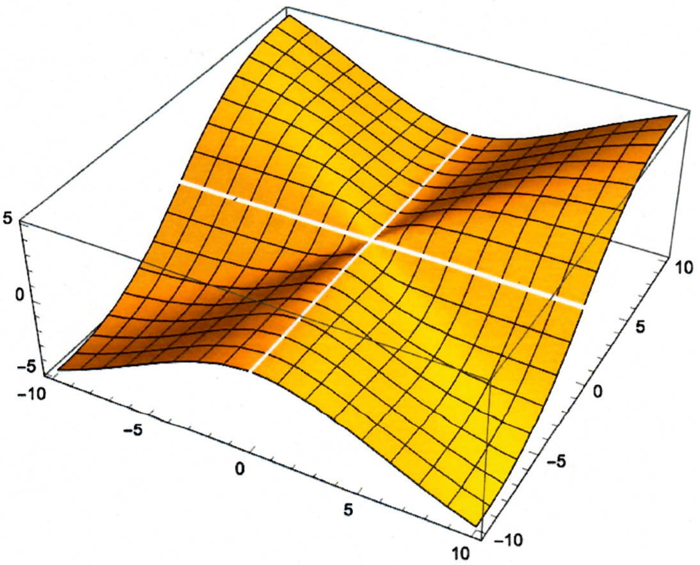
$$\lim_{(r\cos\theta,\, r\sin\theta)\to(0,0)} \frac{r^2\cos^2\theta \, r\sin\theta}{r^2}$$

$$= \lim_{r\to 0} r\cos^2\theta \sin\theta$$

$= 0$ no matter what θ is as we approach $(0,0)$ since $r \to 0$.

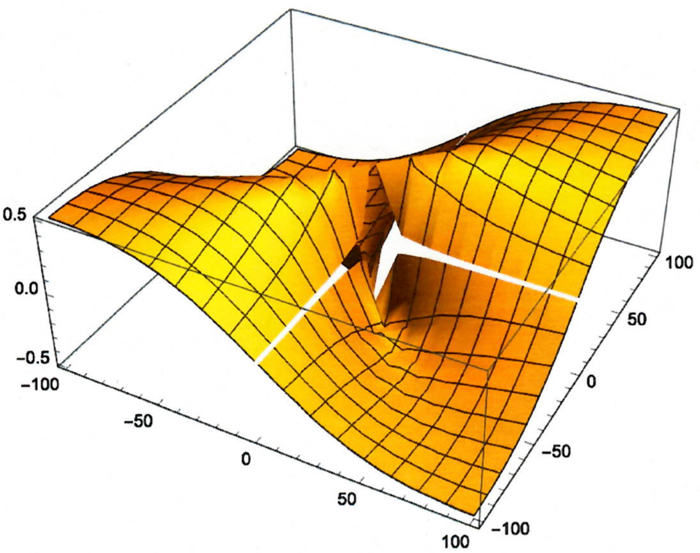
```
In[30]:= Plot3D[x^2*y/(x^2+y^2), {x, -10, 10}, {y, -10, 10}, Exclusions → {x == 0, y == 0}]
```

Out[30]=




```
In[39]:= Plot3D[x*y/(x^2+y^2), {x, -100, 100}, {y, -100, 100}, Exclusions → {x == 0, y == 0}]
```

Out[39]=



For studying functions

$f: \mathbb{R}^n \to \mathbb{R}$

the following lemma can be helpful

Lemma (Sandwich lemma)

If f, g, h are functions $\mathbb{R}^n \to \mathbb{R}$ where

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in \mathbb{R}^n$$

let $a \in \mathbb{R}^n$. If $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then $\lim_{x \to a} g$ also exists and is equal to L.

Eg: $f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} \\ 0 \end{cases}$

$$\left| \frac{x^2}{x^2+y^2} \right| \leq 1$$

$$0 \leq |f(x,y)| \leq |y|.$$

$\lim_{(x,y) \to (0,0)} 0 = 0$ $\quad \lim_{(x,y) \to (0,0)} y = 0$

Clicker

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$y = mx$

$$\lim_{x \to 0} \frac{x^2(mx)}{x^3 + m^3 x^3} = \frac{m}{1+m^3}$$

depends on m

hence the limit doesn't exist.

$$\lim_{r \to 0} \frac{(r\cos\theta)^2 r\sin\theta}{r^3(\cos^3\theta + \sin^3\theta)} = \lim_{r \to 0} \frac{\cos^2\theta \sin\theta}{\cos^3\theta + \sin^3\theta}$$

depends on θ

hence the limit doesn't exist!

Another difficulty w/ many variables is that in dimensions ≥ 2 we also have many options for sets to define these functions hence also sets of "discontinuities"!

Ex: 1) $\cos(x^2 y) = f(x,y)$ defined for all $\mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}$

2) $f : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto \log(x^2+y^2)$

f is defined $\forall\, \mathbb{R}^2$ <u>except</u> $(0,0)$

$\mathbb{R}^2 \setminus \{(0,0)\}$.

3) $\log(\cos(x^2+y^2))$.

not defined if $\cos(x^2+y^2)=0$

when $x^2+y^2 = \left(\frac{2k+1}{2}\right)\pi$