20-10-22

Defn: Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$. Let $y\in\mathbb{R}^n$. $\left(\begin{array}{l} y=(y_1,\dots,y_n)\in\mathbb{R}^n \\ x_k=(x_{k,1},\dots,x_{k,n})\in\mathbb{R}^n\end{array}\right)$

We say (x_k) converges to y if

$\forall\, \varepsilon>0,\ \exists\, N\geq 1$ such that $\forall\, k\geq N$ we have $\|x_k-y\|<\varepsilon$.

We write $\lim_{k\to\infty} x_k = y$ or $x_k \xrightarrow{k\to\infty} y$

Prop: $\lim x_k = y \iff$ For each i, the sequence $(x_{k,i})\in\mathbb{R}$ converge to y_i

ie $(x_{k,i})_k \longrightarrow y_i \quad 1\leq i\leq m$

$\iff$ The sequence of real numbers $\|x_k-y\|$ converge to 0.

Defn. Let $f:X\to\mathbb{R}^m$, $X\subset\mathbb{R}^n$, $x_0\in X$, $y\in\mathbb{R}^m$. We say that f has a limit y, as $x\to x_0$ if $\forall\,\varepsilon>0,\ \exists\,\delta>0$ s.t $\forall\, x\in X,\ x\neq x_0$ such that $\|x-x_0\|<\delta$, we have $\|f(x)-y\|<\varepsilon$. We write $\lim_{x\to x_0} f(x)=y$

Prop: $\lim_{x\to x_0} f(x)=y \iff \forall$ sequence $(x_k)_k\subset X$ such that $\lim x_k = x_0$, we have $\lim_{k\to\infty} f(x_k)=y$.

Defn: We say $f:X\to\mathbb{R}^m$ is continuous at $x_0\in X$ if $\lim_{x\to x_0} f(x)$ exists and is equal to $f(x_0)$.

Prop f is cont. at $x_0 \iff \forall$ sequence (x_k) in X s.t $\lim x_k = x_0$ we have $\lim f(x_k)=f(x_0)$

ie $\boxed{\lim f(x_k) = f(\lim x_k)}$

1

Examples

1) $f: \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$

linear functions are continuous $\forall x \in \mathbb{R}^m$

$p: \mathbb{R}^n \to \mathbb{R}$.

2) Polynomials are continuous $\forall x \in \mathbb{R}^n$

3) If $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^m$ are continuous at x_0 then $f \pm g$ are continuous at x_0

4) If $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$ are continuous then $f \cdot g$, f/g are continuous $\forall x \in \mathbb{R}^n$, s.t $g(x) \neq 0$

5) Functions of separated variables are continous $\Leftrightarrow$ Each factor is continous.

6) $f: \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto (f_1(x), \dots, f_m(x))$ is continuous $\Leftrightarrow$ For each i
$f_i(x): \mathbb{R}^n \to \mathbb{R}$ is continuous.

7) $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^s$ continuous then $g \circ f: \mathbb{R}^n \to \mathbb{R}^s$ is continuous.

8) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} xy/x^2+y^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$ Does not exist

Using polar coordinates gives

$$\lim_{r\to 0} \frac{r^2 \cos\theta \sin\theta}{r^2} = \lim_{r\to 0} \cos\theta \sin\theta$$
$$= \cos\theta \sin\theta$$

<u>Recall</u> Min-Max theorem

<u>Thm</u> Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b)$. Then it attains its max and min. ie $\exists\ x_M, x_m$ so that

$$f(x_M) \geq f(x) \quad \forall x \in [a, b]$$

$$f(x_m) \leq f(x) \quad \forall x \in [a, b]$$

For the analog statement in many variables, we need the analog of compact interval $[a, b]$.

<u>Defn</u> 1) A set $X \subseteq \mathbb{R}^n$ is <u>bounded</u> if the set $\{\|x\| \mid x \in X\}$ is bounded in $\mathbb{R}$.

2) A set $\cancel{X} \subset \mathbb{R}^n$ is called <u>closed</u> if every sequence $(x_k) \subset X$ that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have that $y \in X$

Ie. limits of sequences in X are also in X

3) A set $X \subset \mathbb{R}^n$ is called compact if it is closed and bounded.

Ex 1) $\emptyset$, $\mathbb{R}^n$ closed.

2) $B_r(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$.

is bounded

for every $x \in B_r(x_0)$

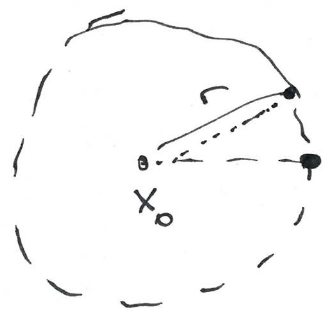
$$\|x\| = \|x - x_0 + x_0\| \le \|x - x_0\| + \|x_0\| = \underbrace{r + \|x_0\|}_{M}.$$

It is not closed

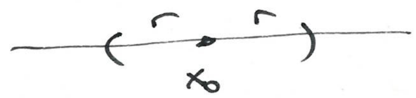
let

$$x_k = x_0 + \left(r - \frac{1}{k}, 0, \dots, 0\right) \in B_{x_0}(r).$$

$$\lim x_k \Longrightarrow x_0 + (r, 0 \dots 0) \notin B_{x_0}(r).$$



$n = 1$ $\quad B_r(x_0) = (x_0 - r, x_0 + r)$



3)

Closed Disc

$$\overline{B_r(x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r\}$$

is closed.

<u>Examples</u> of how to construct closed sets from other closed sets.

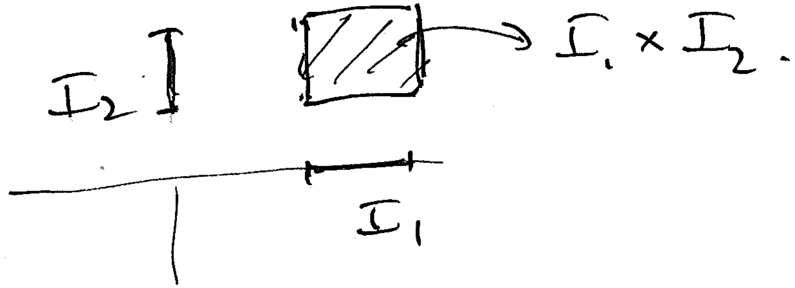
If $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ are bounded (resp closed resp compact) then $X \times Y \subseteq \mathbb{R}^{n+m}$ is bdd (resp closed, compact).

In particular the product

$$I_1 \times I_2 \times \dots \times I_n = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid x_i \in I_i\}.$$

(I_i is an interval in $\mathbb{R}$)

of closed, bounded intervals is also closed and bdd.



2) Using basic examples of closed sets are con construct other closed sets using continuous functions and the following thm.

Thm Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuous. Then for every $Y \subseteq \mathbb{R}^m$ closed, the set $f^{-1}(Y)$ is closed.

Example: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$X = \{x \in \mathbb{R}^n \mid a \leq f(x) \leq b\}$$

$$= f^{-1}([a,b])$$ is closed.

Example $= \{(x,y,z) \in \mathbb{R}^3 \mid \cos(x^3 + e^{xy} + xyz) = 1\}$

Is closed because it is the inverse image of the point $\{1\}$.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longrightarrow \cos(x^3 + e^{xy} + xyz)$$

Thm Let $X \subseteq \mathbb{R}^n$ compact. Let $f: X \to \mathbb{R}$ a continuous function. Then f is bounded and achieves its min and max.

ie $\exists$ $x_+, x_- \in X$ s.t

$$f(x_+) = \sup_{x \in X} f(x)$$

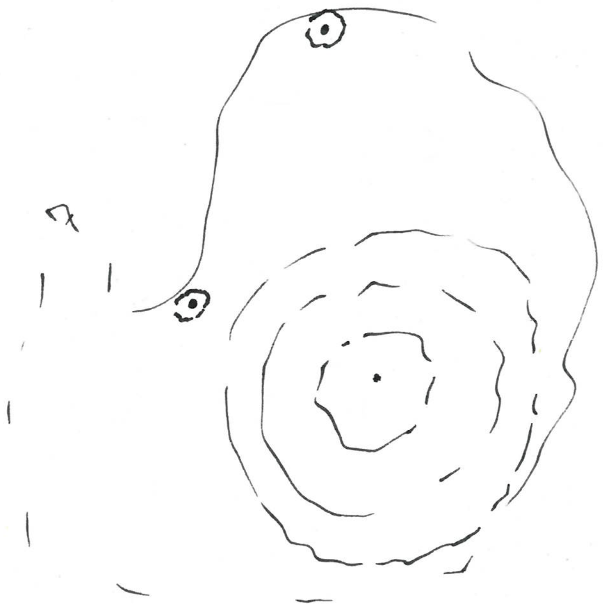
$$f(x_-) = \inf_{x \in X} f(x).$$

We also have a defn of open sets.

Defn: $X \subseteq \mathbb{R}^n$ is called open if its complement in $\mathbb{R}^n$, $\mathbb{R}^n \setminus X$, is closed.

This is equivalent to

$\forall x \in X$, $\exists r > 0$ s.t the set $\{y \in \mathbb{R}^n \mid \|y - x\| < r\}$
$= B_r(x) \subset X$



Clicker $f : \mathbb{R}^2 \to \mathbb{R}$

s.t. along any line

$y = mx$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} f(x,mx)$

$= L$

Then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$

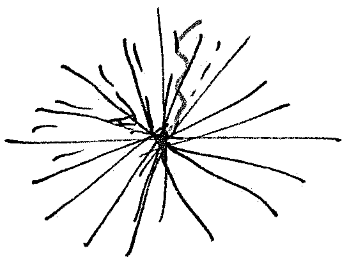
False.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (0,0). \end{cases}$$

$$\lim_{(x,mx)\to(0,0)} f(x,mx)$$

$$= \lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = 0.$$

$$\lim_{x\to 0} \frac{mx}{x^2+m^2}$$



Approach along $y = x^2$

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}.$$

§3.3 Partial Derivatives

$$f : \mathbb{R} \to \mathbb{R}^m$$
$$x \mapsto (f_1(x), \ldots, f_m(x))$$

$$f_i(x) : \mathbb{R} \to \mathbb{R}.$$

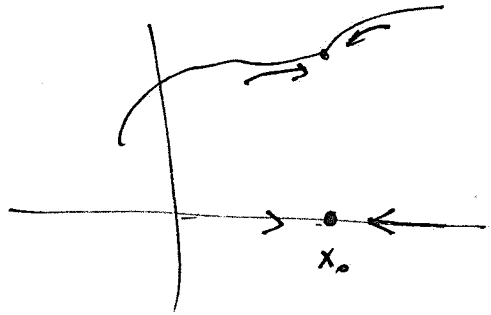
In this case we say f is differentiable at x_0 if and only if each $f_i : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0.

In this case

$$f'(x_0) = (f_1'(x_0), \ldots, f_m'(x_0)).$$

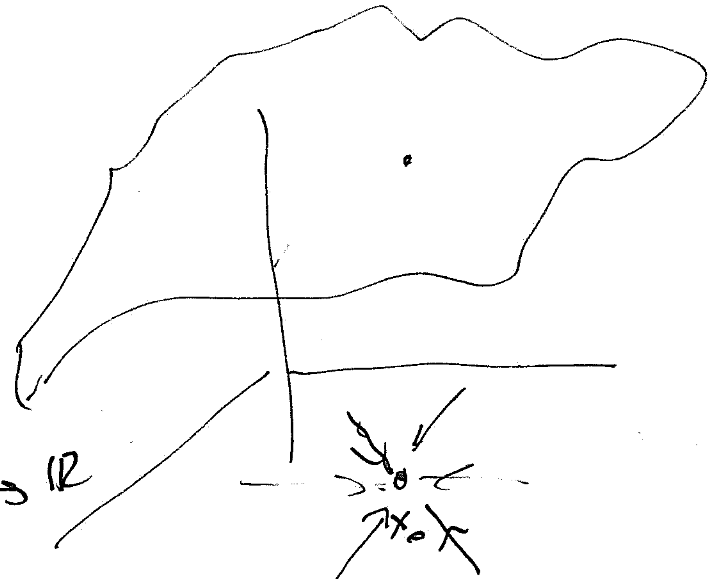
Any function
$$f : \mathbb{R}^n \to \mathbb{R}^m$$
is also a collection of functions
$$f_i : \mathbb{R}^n \to \mathbb{R}$$
$$1 \le i \le m.$$

We restrict ourselves to functions $f : \mathbb{R}^n \to \mathbb{R}$.

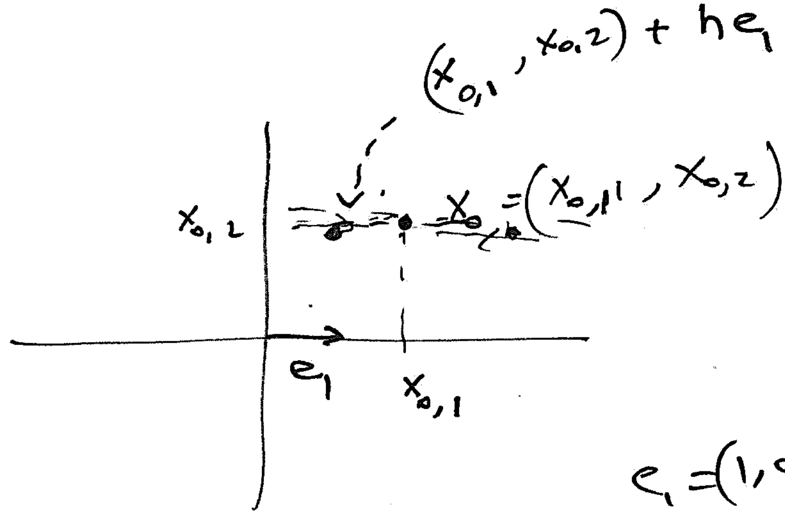


$f : \mathbb{R} \to \mathbb{R}$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



$f : \mathbb{R}^2 \to \mathbb{R}$



$e_1 = (1,0)$

<u>$n=2$</u> case.

We can approach $x_0 = (x_{0,1}, x_{0,2})$ in the horizontal direction

i.e. I fix the second component and look at

$$\frac{\Delta f}{\Delta x_1} = \frac{f(x_{0,1}+h, x_{0,2}) - f(x_{0,1}, x_{0,2})}{h}.$$

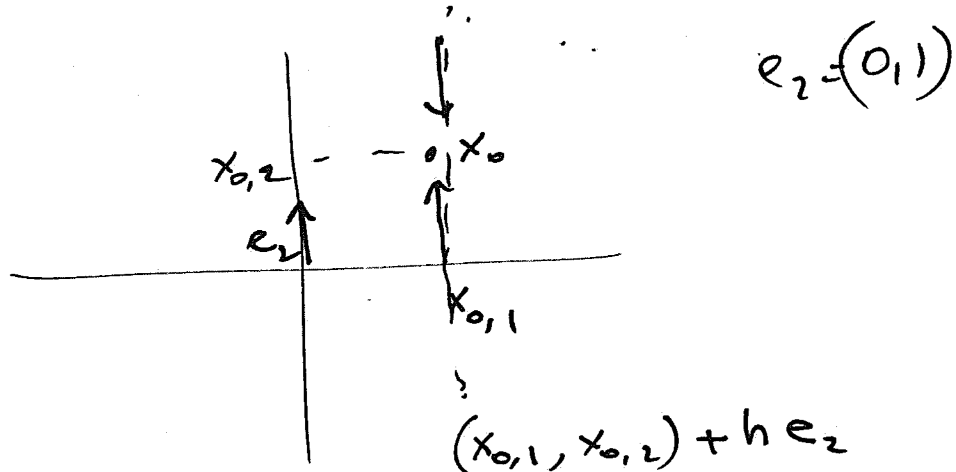
$\Delta x_1 \to$ the change in the first component

Take the limit as $h \to 0$.

$$\lim_{h \to 0} \frac{f(x_{0,1}+h,\ x_{0,2}) - f(x_{0,1},\ x_{0,2})}{h}$$

If it exists we call it the partial derivative of f with respect to the first component x_1 at the point x_0.

Denote it by $\dfrac{\partial f}{\partial x_1}(x_0)$



$$\lim_{h \to 0} \frac{f(x_{0,1},\ x_{0,2}+h) - f(x_{0,1},\ x_{0,2})}{h}.$$

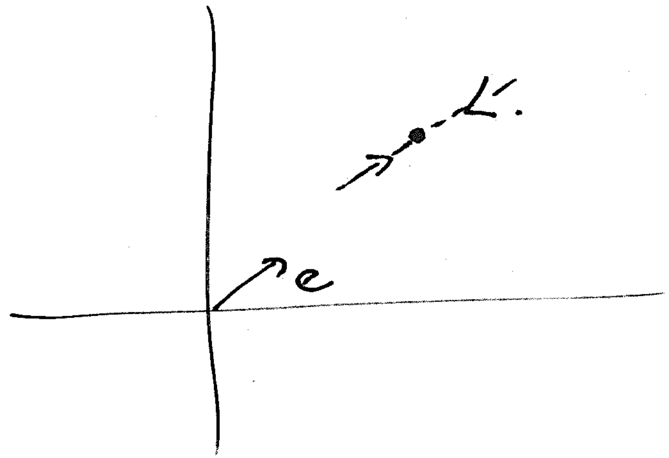
$$=: \frac{\partial f}{\partial x_2}(x_0).$$

Rk What we are really doing is the following: We are considering the function

$$g(t) := f(x_{0,1}, t)$$

and looking at the 1-variable function's derivative in t at $t = x_{0,2}$.

We could approach also along any other direction.



$$x_0 + he = (x_{0,1}, x_{0,2}) + he$$

$$\lim \frac{f(x_0 + he) - f(x_0)}{h}.$$

$\nearrow$ directional derivative of f at x_0, in the direction of e.

$f : \mathbb{R}^2 \to \mathbb{R}$.

$\operatorname{graph} f = \{(x, y, z) \mid z = f(x,y)\}$.

$z_0 = f(x_0, y_0)$

$P = (x_0, y_0, z_0)$

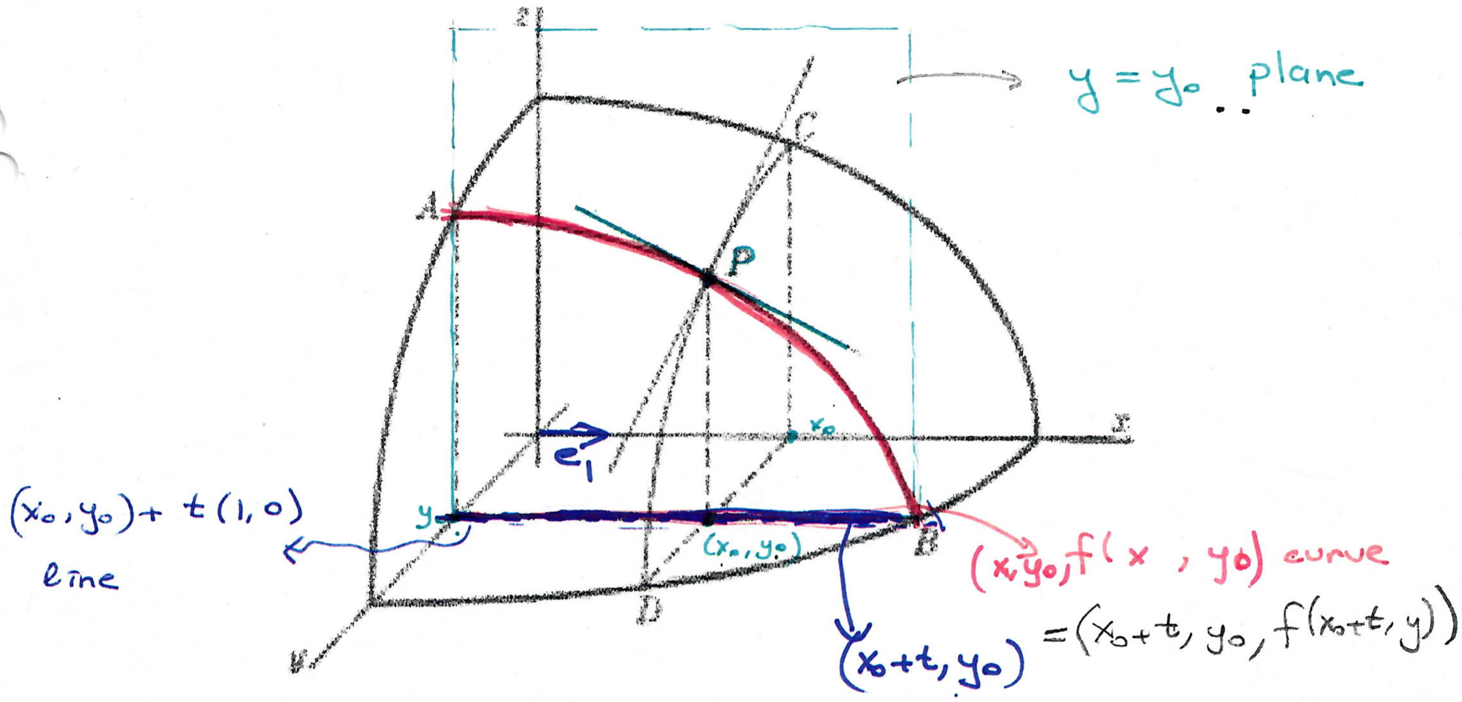
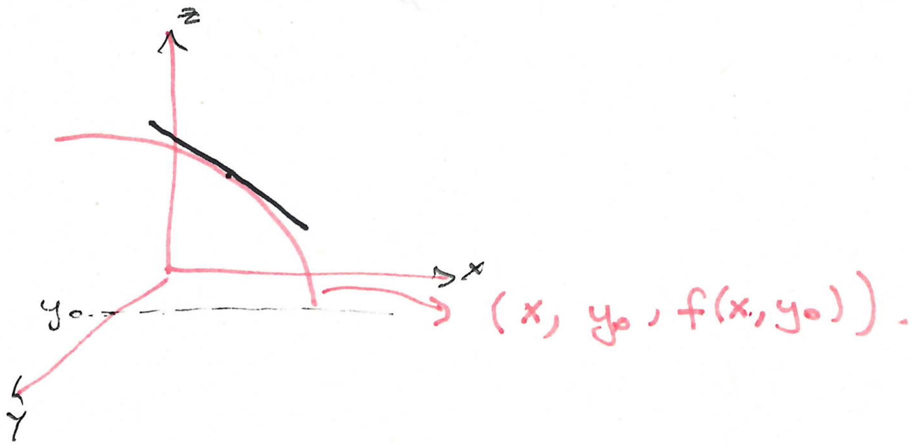


Fig. 1

$$\text{slope} = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dt}\left(f(x_0+t, y_0)\right)\Big|_{t=0}.$$



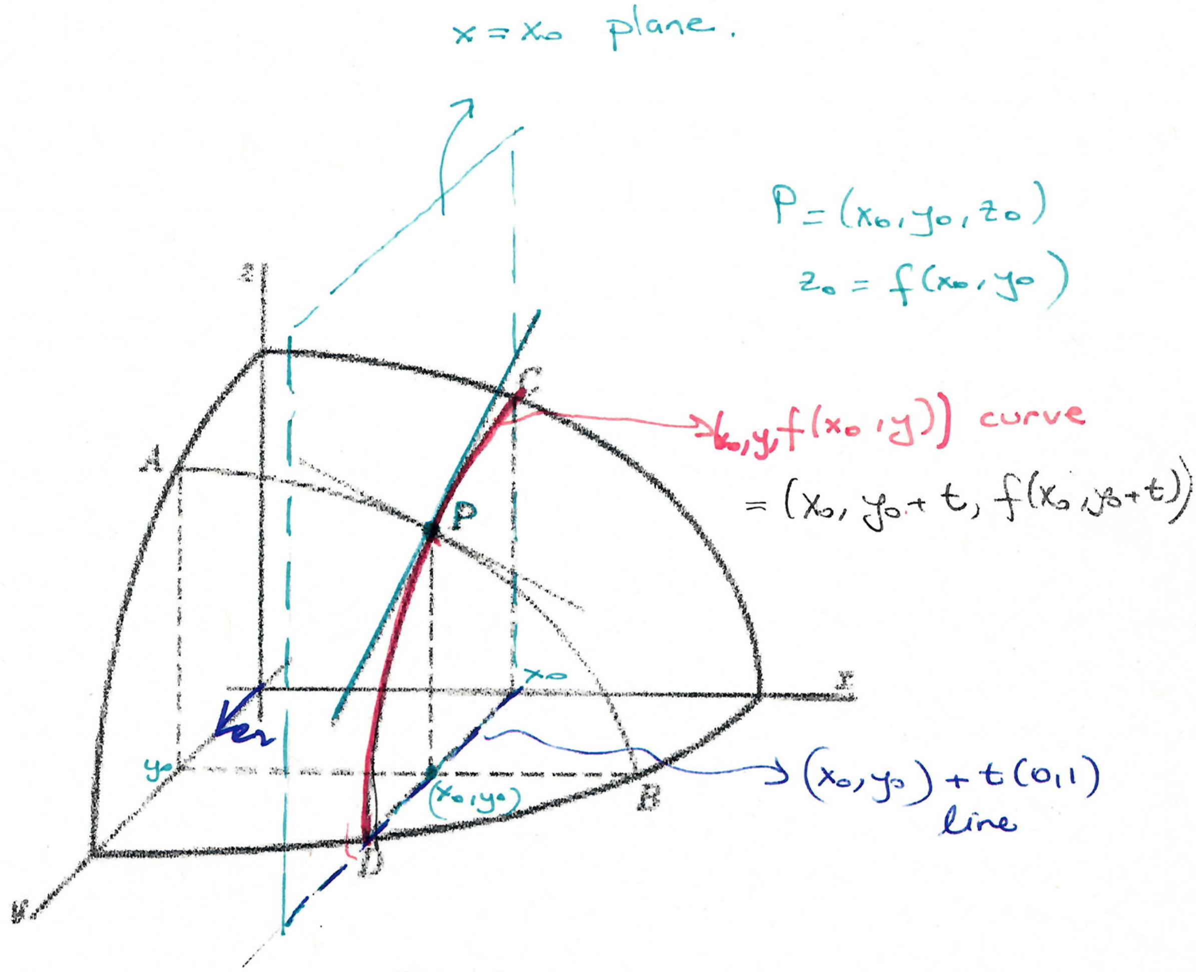
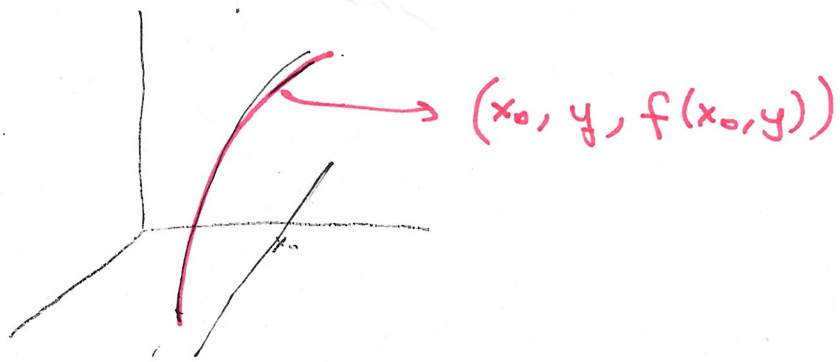


Fig. 1

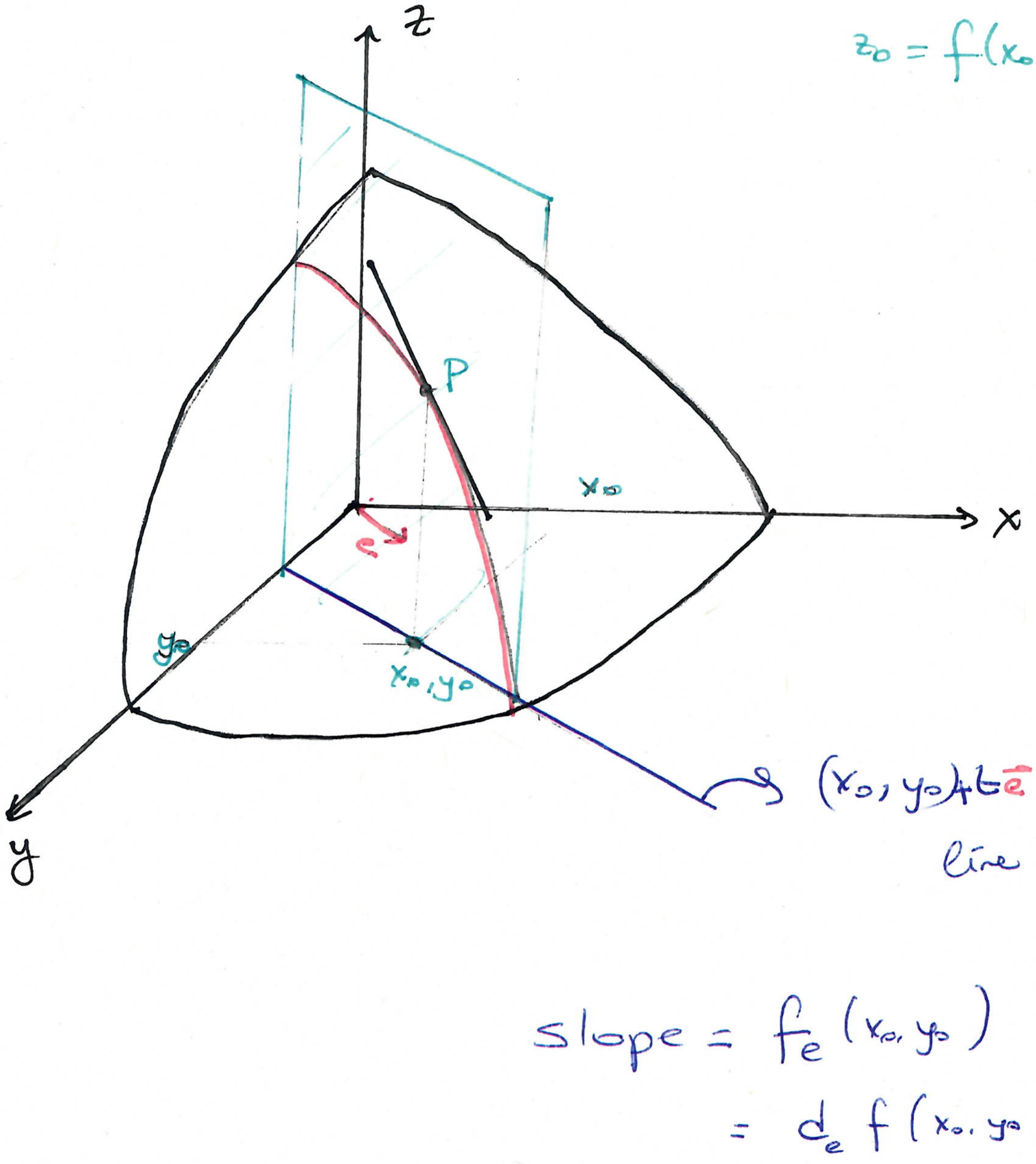
$$\text{slope} = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dt}\left(f(x_0, y_0 + t)\right)\Big|_{t=0}.$$



In any direction $\underline{e}$

$P = (x_0, y_0, z_0)$

$z_0 = f(x_0, y_0)$



$$\text{slope} = f_e(x_0, y_0)$$

$$= d_e f(x_0, y_0)$$

$$= \frac{d}{dt} f\big((x_0, y_0) + t\vec{e}\big)\Big|_{t=0}$$

If $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

and we want to study

f around $x_0 = (x_{0,1}, x_{0,2}, \dots x_{0,n})$

For each j, we consider

the 1-variable function

$$g_j(t) := f(x_{0,1}, x_{0,2}, \dots x_{0,j-1}, t, x_{0,j+1}, \dots, x_{0,n})$$

defined on the set $A = \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots x_{0,n}) \in X\}$.

where we fixed all variables except the j-th one.

$$\frac{dg_j}{dt}(x_{0,j}) = \lim_{h \to 0} \frac{g(x_{0,j} + h) - g(x_{0,j})}{h} =$$

$$\frac{dg_j}{dt}(x_{o,j}) =$$

$$\lim_{h \to 0} \frac{f(x_{o,1}, \ldots x_{o,j-1}, (x_{o,j}+h), x_{o,j+1}, \ldots x_{on}) - f(X_o)}{h} = \frac{\partial f}{\partial x_j}(x_o)$$

and ask if this limit exists.

If it does then we say

f has partial derivative with respect to x_j at the point x_o, and for the limit we write

$\frac{\partial f}{\partial x_j}(x_o)$ $x_o \to$ fixed point. $\partial x_j \to$ j-th variable, $(\partial_{x_j} f)(x_o)$, $(\partial_j f)(x_o)$

To evaluate partial derivatives of a function

$f: \mathbb{R}^n \to \mathbb{R}^m$

with respect to one of the variables ~~...~~ say, x_2

differentiate $f(x_1, x_2, \dots, x_n)$ with respect to x_2 treating all other variables $x_1, x_3, \dots x_n$ as a constant wrt x_2.

Ex. 1) $f: \mathbb{R}^2 \to \mathbb{R}$

$(x, y) \to (x^2 + xy) \sin y$.

$$\frac{\partial f}{\partial x}(x, y) = \sin y \,(2x + y)$$

$$\frac{\partial f}{\partial y}(x, y) = x \sin y + (x^2 + xy) \cos y.$$

2) $f: \mathbb{R}^2 \to \mathbb{R}^3$

$(x, y) \mapsto (x^2 + y^2, 2x, 2y)$

$$\frac{\partial f}{\partial x} = (2x, 2, 0)$$

$$\frac{\partial f}{\partial y} = (2y, 0, 2).$$

3) $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x, y, z) \to x^3 y z^2 + \cos x + z^2$$

$$\frac{\partial f}{\partial x} = 3x^2 y z^2 - \sin x$$

$$\frac{\partial f}{\partial y} = x^3 z^2 \qquad \frac{\partial f}{\partial z} = 2x^3 y z + 2z$$

4) $f(x_1, \dots x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n.$

$$\frac{\partial f}{\partial x_j} = a_j$$

5) $f : \mathbb{R}^n \to \mathbb{R}^m$

$$x \to Ax$$

$A_{m \times n}$ matrix.

$$\frac{\partial f}{\partial x_j}(x) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$Ax = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

$$f_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

$$1 \le i \le m$$

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

$$\frac{\partial f_i}{\partial x_j} = a_{ij}$$

We can put all these derivatives of f_i's wrt x_j's into a matrix

$$J_f := \left(\frac{\partial f_i}{\partial x_j}(x)\right) = A$$

<u>Defn</u> 1) Let $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ a function with partial derivatives on X

Write $f(x) = (f_1(x), \ldots, f_m(x))$

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

For any $x \in X$ the matrix

$$J_f(x) := \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

with m rows, n columns is called the <u>Jacobi Matrix</u> of f.

2) if $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

if all partial derivatives exist then the column vector $\begin{pmatrix} \frac{\partial f(x_0)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x_0)}{\partial x_n} \end{pmatrix}$ is called the gradient of f of x_0, and is denoted by $\nabla f(x_0)$.

Note $\nabla f = \left(J_f(x_0) \right)^t$

$$= \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)^t.$$

Prop.

1) If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ have partial derivatives wrt x_j then so does $f+g$, $f-g$

$$\frac{\partial (f+g)}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j}$$

2) $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$

f, g have partial derivatives wrt x_j then so does fg

$$\frac{\partial (fg)}{\partial x_j} = \left(\frac{\partial f}{\partial x_j} \right) g + f \frac{\partial g}{\partial x_j}$$

Similarly for f/g
for $g(x_0) \neq 0$.

$$\frac{\partial (f/g)}{\partial x_j}(x_0) = \frac{\frac{\partial f}{\partial x_j}(x_0) \cdot g(x_0) - \frac{\partial g}{\partial x_j}(x_0) f(x_0)}{g^2(x_0)}.$$

3) if $f : X \longrightarrow \mathbb{R}^m$
$X \subseteq \mathbb{R}^n$, ~~and~~ then
$\partial_{x_i} f$ are themselves
functions of n variables

and can be partial
differentiated to obtain
higher order partial,
derivatives.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \partial_{x_j, x_i} f$$
$$= \partial_{ji} f$$

Ex $f(x,y,z) = x^3 y z^2 + \cos x + z^2$

$$\frac{\partial f}{\partial x} = 3x^2 y z^2 - \sin x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3x^2 z^2, \quad \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = 6x^2 z$$

Ex: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \left(\frac{f(0, 0+h) - f(0,0)}{h} \right)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{\frac{h \cdot 0}{h^2+0^2} - 0}{h} \right).$$

$$= \lim_{h \to 0} \frac{0}{h} = 0.$$

$= 0.$

This shows that existence of partial derivatives is not strong enough.

(lim f does not exist hence f is not cont at $(0,0)$).