24.10.22

- $f: \mathbb{R}^n \to \mathbb{R}$

The partial derivative of f with respect to the i-th variable at a point $a = (a_1, \dots, a_n)$, denoted by

$$\frac{\partial f}{\partial x_i}(a) \quad \text{or} \quad \partial_i f(a) \quad \text{is}$$

defined as

$$\boxed{\frac{\partial f}{\partial x_i}(a) = \lim_{h \to 0} \frac{f(a+he_i) - f(a)}{h}}$$

where $e_i = (0, \dots 0, 1, 0, \dots 0)$ is the i-th canonical Basis vektor of $\mathbb{R}^n$.

Equivalently: let $g(t) = f(a+te_i)$

$$g: \mathbb{R} \to \mathbb{R}$$
$$t \mapsto f(a+te_i)$$

Then $\boxed{\frac{\partial f}{\partial x_i}(a) = g'(0)}$.

- For $f: \mathbb{R}^n \to \mathbb{R}^m$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (f_1(x), \dots, f_m(x))^t$$

$$\boxed{\frac{\partial f}{\partial x_i}(a) := \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(a) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(a) \end{pmatrix}}$$

- For any $v \in \mathbb{R}^n$ the directional derivative of f at a in the direction of v is similarly defined as

$$\boxed{g'(0) =: \partial_v f(a) \quad \text{where} \quad g(t) := f(a+tv).}$$

$$\partial_v f(a) := \lim_{h \to 0} \frac{f(a+hv) - f(a)}{h}.$$

Defn (Jacobi Matrix)

$f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

For $x \in X$, The Jacobi Matrix of f at $x \in X$ is the $m \times n$ matrix

$$\boxed{J_f(x) := \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}.}$$

(Assuming: All partial derivatives exist).

Special case:

$f : \mathbb{R}^n \longrightarrow \mathbb{R}.$

$$J_f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots \frac{\partial f}{\partial x_n}(x) \right).$$

$\nabla f(x)$ = the Gradient of f at x

is the Vektor

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} =: \nabla f(x)$$

read as Nabla

$\underline{\text{Ex}}$ $$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = 0.$$

$\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does not exist, hence f is not continous.

Hence

$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exists at a point x

$\not\Rightarrow$ f is continuous at x !

We need a stronger differentiability Criteria !

Recall :

$f : \mathbb{R} \longrightarrow \mathbb{R}$

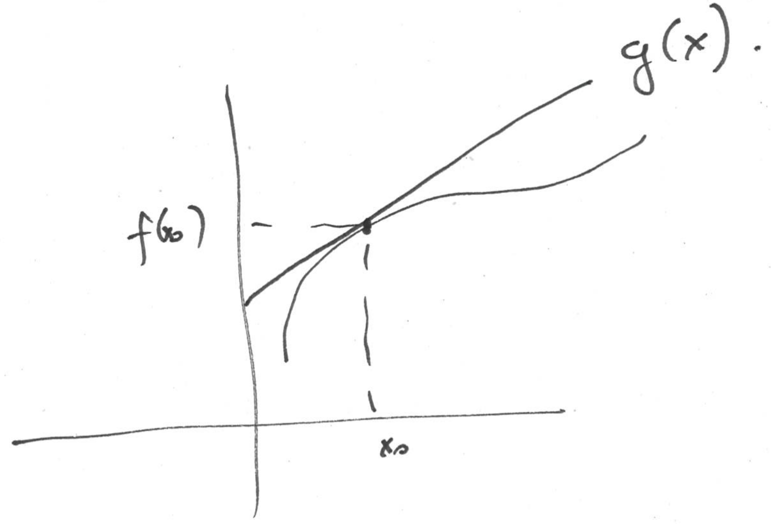
the existence of the derivative at a gien point x_0 says : the function f can be well approximated by the affine linear map.

$$g(x) = f(x_0) + \overbrace{f'(x_0)(x-x_0)}^{u(x-x_0)}$$

well approximated means :

if $f(x) = g(x) + R(x, x_0)$

$$= f(x_0) + f'(x_0)(x-x_0) + R(x, x_0)$$



then $\dfrac{R(x, x_0)}{|x - x_0|} \longrightarrow 0$ as $x \to x_0$

Goal: To show that for $f : \mathbb{R}^n \to \mathbb{R}^m$ this is the right way to look at differentiability.

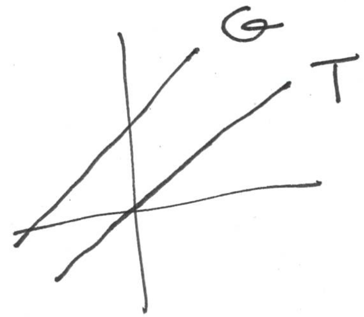
We want to say that f is diff at a point $x_0 \in \mathbb{R}^n$, if there is an affine linear map which approximates it well.

$$T : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow a x$$

is linear.

$$G : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto ax + b$$

is affine linear.



Recall: Linear Algebra

For a linear map

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

we can always find a matrix repn. of T once we fix Bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

ie. $\exists\ A \in \mathbb{R}^{m\times n}$

Matrix such that

$T(x) = Ax$

In case that $m=1$, $n=1$

$T : \mathbb{R} \to \mathbb{R}$
$x \to ax$

We can view "a" as
1x1 matrix representation
of T.

$f : \mathbb{R} \to \mathbb{R}$

f is diff at x_0

if $f(x) = f(x_0) + \boxed{f'(x_0)(x-x_0)}$

$+$ Error

with $\dfrac{\text{Error}}{x - x_0} \to 0$ $x \to x_0$.

ie $\exists$ a linear map
$u : \mathbb{R} \to \mathbb{R}$
$x \to ax$

with $a = f'(x_0)$

$f(x) = f(x_0) + u(x-x_0) + \text{Error}$

Defn let $X \subset \mathbb{R}^n$ open, $x_0 \in X$

$f : X \to \mathbb{R}^m$ a functi.

We say f is differentiable at x_0, with differential u

if there exists a linear map $u : \mathbb{R}^n \to \mathbb{R}^m$

such that

$$\boxed{\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - (f(x_0) + u(x-x_0))}{\| x - x_0 \|} = 0}$$

limit in $\mathbb{R}^m$.

ie. the affine linear map

$$g(x) = f(x_0) + u(x - x_0)$$

approximates f well

ie. error goes to zero faster than $\|x - x_0\|$

The linear map

$u : \mathbb{R}^n \to \mathbb{R}^m$!

is called (total) differential of f at x_0; denoted by $df(x_0)$, $d_x f$

Remark For $f: \mathbb{R}^n \to \mathbb{R}^m$

the differential $df(x_0)$

is **NOT** a number

but **IS** a **linear map**

The linear map $u = df(x_0)$

depends on the point x_0.

We have a different

linear map for each

point x_0.

Just like for $f: \mathbb{R} \to \mathbb{R}$

the linear map

$$u : \mathbb{R} \to \mathbb{R}$$
$$x \to \underbrace{f'(x_0)}_{a} x$$

changes as we change

the point x_0.

Question: We know every

linear map $u : \mathbb{R}^n \to \mathbb{R}^m$

has a Matrix repn

So if f is differentiable at x_0

It means $\exists$ a lin. map $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$

What is the matrix representation of this linear map?

Do we recover the theorem that

differentiable $\Rightarrow$ continuous?

Yes!

Thm $f : \Omega \in \mathbb{R}^n \to \mathbb{R}^m$ be diff. at x_0.

① f is continuous at x_0.

② f has all partial derivatives at x_0.

③ The matrix that represents the differential in the standard basis is

$$J_f(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

In the special case that $m = 1$

$f : \mathbb{R}^n \to \mathbb{R}$

$df(x_0) : \mathbb{R}^n \to \mathbb{R}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x \longmapsto \underbrace{\left(\frac{\partial f}{\partial x_1}(x_0), \dots \frac{\partial f(x_0)}{\partial x_n} \right)}_{A_{1 \times n}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

In terms of the gradient

$$d_{x_0}f(v) = \nabla f(x_0) \cdot v$$

scalar product.

$v = (v_1, v_2, \ldots, v_n)$

$$\frac{\partial f}{\partial x_1}(x_0)\, v_1 + \frac{\partial f}{\partial x_2}(x_0)\, v_2 + \cdots + \frac{\partial f}{\partial x_n}(x_0)\, v_n.$$

Proof of (1).

Assume f is diff. at x_0

Then $f(x) = f(x_0) + \overbrace{d_{x_0}f(x-x_0)}^{u(x-x_0)} + R(x, x_0)$ (*).

where $\dfrac{R(x, x_0)}{\|x - x_0\|} \xrightarrow[x \to x_0]{} 0$.

In particular. $\lim_{x \to x_0} R(x, x_0) = 0$.

$$\lim_{x \to x_0} u(x - x_0) = u(0) = 0$$

$\downarrow$

u is a continuous map being a linear map.

If we take the limit on both sides of (*) as $x \to x_0$.

$$\lim_{x \to x_0} f(x) = f(x_0) + \overbrace{\lim_{x \to x_0} u(x - x_0)}^{0} + \underbrace{\lim_{x \to x_0} R(x, x_0)}_{0}$$

$\Rightarrow f$ is continuous

Recall $f: \mathbb{R} \to \mathbb{R}$ diff in x_0 w/ $f'(x_0) = a$

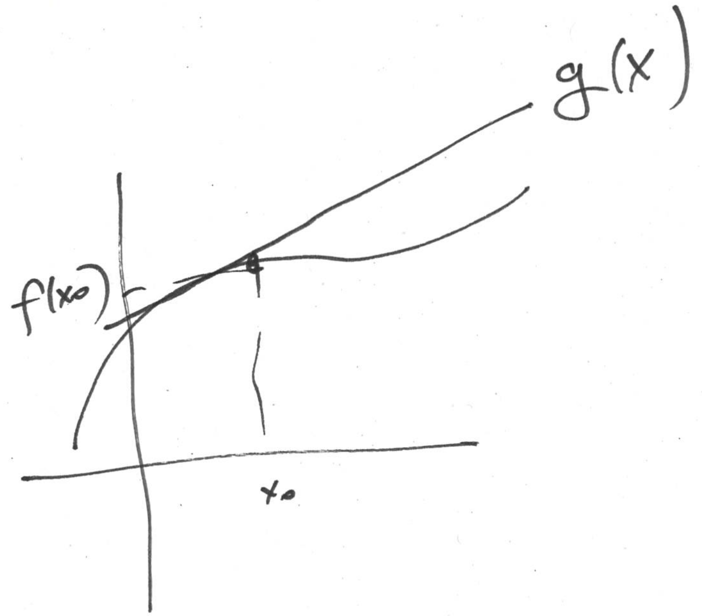
Graph of the linear / affine map

$$g(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$f(x_0) + u(x - x_0)$$

where

$$u : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f'(x_0)x$$

$g(x)$ is the linear approx. to f, which is the tangent line at x_0 to the graph of f



x

∫

Similarly we define

Defn: Let $X \subseteq \mathbb{R}^n$ $f: X \to \mathbb{R}^m$ differentiable at x_0 with differential

$$d_{x_0} f = u : \mathbb{R}^n \to \mathbb{R}^m$$

The graph of the affine linear map

$$g(x) = f(x_0) + u(x - x_0)$$

is called the tangent space at x_0 to the graph of f

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}.$$

$$= \{(x, g(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid g(x) = f(x_0) + u(x - x_0)\}.$$

Ex if $f: \mathbb{R}^2 \to \mathbb{R}$ diff $(a,b) \in \mathbb{R}^2$

$$d_{(a,b)} f : \mathbb{R}^2 \to \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(d_{(a,b)} f)\begin{pmatrix} x \\ y \end{pmatrix} = A_1 x + A_2 y$$

where $A_1 = \frac{\partial f}{\partial x}(a,b)$ $A_2 = \frac{\partial f}{\partial y}(a,b)$

$$g(x,y) = f(a,b) + (\nabla f)(a,b) \cdot (x-a, y-b)$$

$$= f(a,b) + A_1(x-a) + A_2(y-b).$$

Graph of g:

$$\{(x, y, g(x,y)) \in \mathbb{R}^3\}.$$

$$\{(x, y, f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0))\}$$

$$\overset{\parallel}{x - x_0 + x_0}$$

$$= \{ (x_0, y_0, f(x_0,y_0)) + (x-x_0, y-y_0, \nabla f(x_0,y_0) \cdot (x-x_0, y-y_0)) \}.$$

Properties of the differential.

① If $f, g : X \subset \mathbb{R}^n \to \mathbb{R}^m$ diff. in x_0 then so is $f+g$

and $d_{x_0}(f+g) = d_{x_0} f + d_{x_0} g$.

$m=1$

② $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$.

If f, g are diff. at x_0 then so is fg

If g is non-zero then also f/g.

$$d_{x_0}(fg) = (d_{x_0} f)\, g(x_0) + f(x_0)\,(d_{x_0} g)$$

We have seen that

f diff $\Longrightarrow$ f continuous

f has all partial derivatives $\not\Longrightarrow$ f continuous

There is a partial converse

Thm If $f : X \to \mathbb{R}^m$ has all partial derivatives

$$\frac{\partial f_i}{\partial x_j} : X \to \mathbb{R}^m$$

and if these functions are continuous on X then f is differentiable on X.

Rk This theorem says that in practice many functions are differentiable.

Ex: ① $f(x,y) = \begin{pmatrix} x^2+y^2 \\ 2x \\ 2y \end{pmatrix}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\frac{\partial f}{\partial x}(x,y) = \begin{pmatrix} 2x \\ 2 \\ 0 \end{pmatrix}$, $\frac{\partial f}{\partial y}(x,y) = \begin{pmatrix} 2y \\ 0 \\ 2 \end{pmatrix}$.

$d_{(x_0,y_0)} f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_0 & 2y_0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$.

at $x_0 = 1$, $y_0 = 2$

$d_{(1,2)} f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

$= \begin{pmatrix} 2t_1 + 4t_2 \\ 2t_1 \\ 2t_2 \end{pmatrix}$.

② $f(x,y,z) = x^3 y z^2 + \cos x + z$.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

$(x_0, y_0, z_0) = (0, 1, 2)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,1,2)} = 3x^2yz^2 - \sin x = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,1,2)} = x^3z^2 = 0$$

$$\left.\frac{\partial f}{\partial z}\right|_{(0,1,2)} = x^3y \cdot 2z + 1 = 1$$

$$d_{(0,1,2)}f : \mathbb{R}^3 \to \mathbb{R}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (0,0,1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$d_{x_0}f(x,y,z) = z$$

Ex: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq 0,0 \\ 0 & \text{o.w.} \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0.$$

If f were differentiable then the differential would be the zero map.

$$u : \mathbb{R}^2 \to \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to (0,0)\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Then we would have

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - u(x,y)}{\|(x,y) - (0,0)\|} = 0.$$

$\Rightarrow$ would mean.

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim \frac{xy}{(x^2+y^2)^{3/2}} = 0.$$

check that

$\lim \frac{xy}{(x^2+y^2)^{3/2}}$ does not exist.