10.11.22

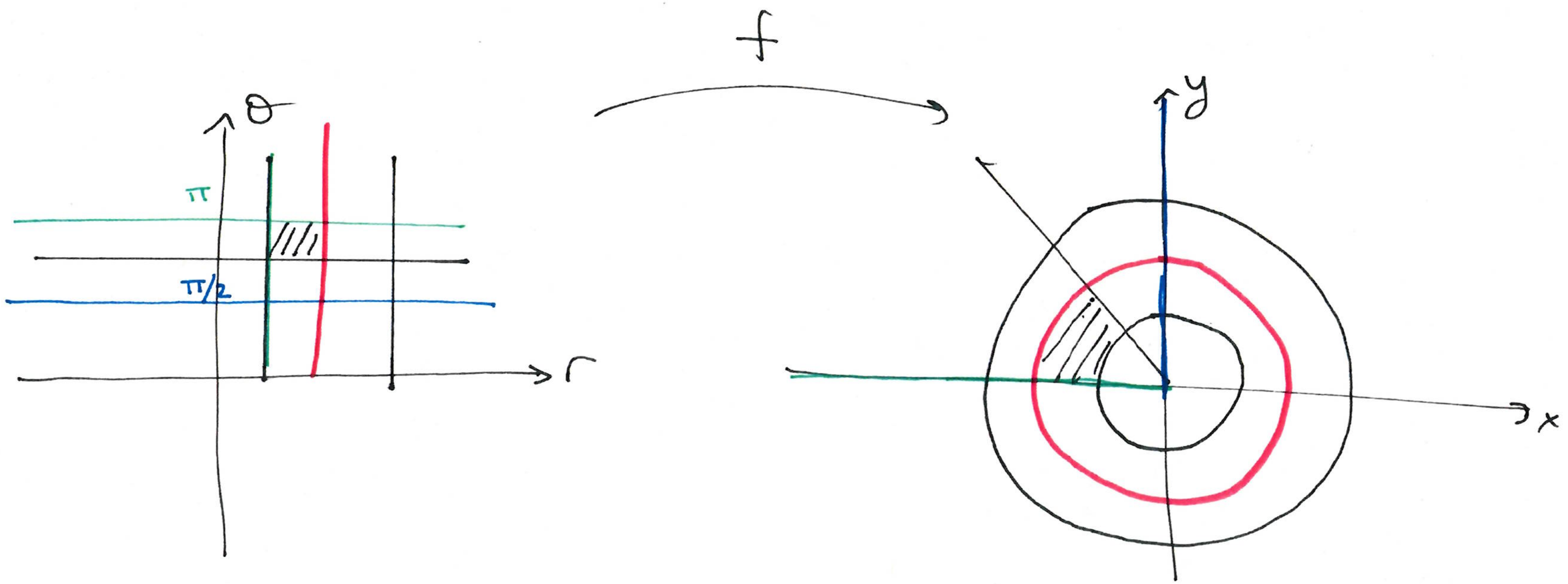
Polar coordinates

$$f : [0, \infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$$

$$(r, \theta) \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} f_1(r,\theta) \\ f_2(r,\theta) \end{pmatrix}.$$

$$J_f(r,\theta) = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

$$\det J_f(r,\theta) = r(\cos^2\theta + \sin^2\theta) = r$$


f
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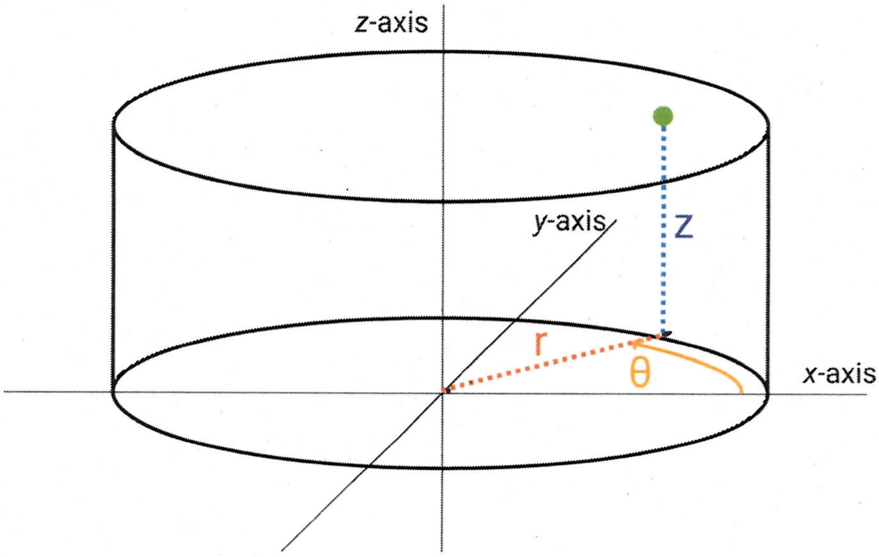


FIGURE 1. Cyclindrical Coordinates

$$f : (0,\infty) \times (0, 2\pi) \times \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$(r, \theta, z) \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$J_f(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det J_f = r$$

/3

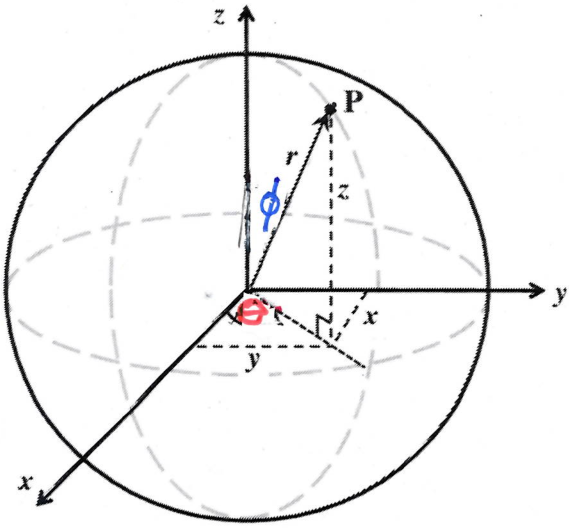


FIGURE 2. Spherical Coordinates

$$f : (0, \infty) \times (0, 2\pi) \times (0, \pi) \longrightarrow \mathbb{R}^3$$

$$(r, \theta, \phi) \longmapsto \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$J_f(r, \theta, \phi) = \begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\phi\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

$$\det J_f = \underline{-r^2\sin\phi}$$

Defn: Change of variables

Let $X \subset \mathbb{R}^n$ be open and $f: X \to \mathbb{R}^n$ differentiable

We say f is a change of variables around $x_0 \in X$ if there is a radius $r > 0$ such that the restriction of f to Ball $B_r(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$.

has the property that the image $Y = f(B_r(x_0))$ is open in $\mathbb{R}^n$ and $\exists$ diff. map $g: Y \to B$ s.t $f \circ g = id$, $g \circ f = id$.

Thm (Inverse function theorem) Let $X \subseteq \mathbb{R}^n$ open, $f: X \to \mathbb{R}^n$ diff. If $x_0 \in X$ is such that $\det(J_f(x_0)) \neq 0$ i.e. $J_f(x_0)$ is invertible then f is a change of variables around x_0

Moreover the Jacobian of g is determined by $J_g(f(x_0)) = J_f(x_0)^{-1}$.

Rk This is the analog of the fact that in $n=1$

for a function
$f: I \to \mathbb{R}$
f is bijektive from I to its image if
$f'>0$ (or $f'<0$).

Higher order partial derivatives

Recall $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$
diff.
the eqn of the tangent ~~plane~~ space gives the first order approximation to f.

$$f(x) \approx \boxed{f(x_0) + \nabla f(x_0) \cdot (x - x_0)}$$

Defn. $X \subset \mathbb{R}^n$ open

$f : X \longrightarrow \mathbb{R}^m$.

We say f is diff. of class C^1 if f is differentiable on X and all its partial derivatives are continuous.

The set of all C^1 functions are denoted by

$C^1(X : \mathbb{R}^m)$

let $k \geq 2$

we say $f \in C^k(X : R^m)$ $\cancel{f}$ or f is of class C^k if its differentiable and each $\partial_{x_i} f : X \longrightarrow \mathbb{R}^m$ $1 \leq i \leq n$ is of class C^{k-1}

We say f is smooth or C^∞ if $f \in C^k$ $\forall$ k.

Rk All polynomials, trig functions exponential function are of class C^∞.

Ex $f(x,y,z) = 3x^2y + 4yz$

$$\frac{\partial f}{\partial x} = 6xy \qquad \frac{\partial f}{\partial y} = 3x^2 + 4z$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6x \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x$$

Ex $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Homework: $\partial_x(\partial_y f)(0,0) = 1$

$\partial_y(\partial_x f)(0,0) = -1$

Thm If $f \in C^k$, $k \geq 2$ then the partial derivatives of order $\leq k$ are independent of the order of differentiation.

For example for $f \in C^2$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

$f \in C^4$

$$\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)\right)$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_k}\right)\right)$$

$= \cdots$

Notation

$$\frac{\partial}{\partial x_{i_k}} \cdots \left(\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f}{\partial x_{i_1}}\right)\right) = \frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}}$$

Defn let $f : X \to \mathbb{R}$

$X \subseteq \mathbb{R}^n$.

If $f \in C^2(X : \mathbb{R})$

$x_0 \in X$. Then the

$n \times n$ matrix $\left(\dfrac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$.

is called the Hessian of f at x_0.

Sometimes it is also denoted by $\nabla^2 f(x_0)$, $\mathrm{Hess}_f(x_0)$.

Example $f : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \mapsto x^2 y + y^z$

$\dfrac{\partial f}{\partial x} = 2xy \qquad \dfrac{\partial f}{\partial y} = x^2 + z \qquad \dfrac{\partial f}{\partial z} = y$

$\dfrac{\partial^2 f}{\partial x^2} = 2y \qquad \dfrac{\partial f}{\partial x \partial y} = 2x \qquad \dfrac{\partial^2 f}{\partial x \partial z} = 0$

$\dfrac{\partial f}{\partial y \partial x} = 2x \qquad \dfrac{\partial f}{\partial z \partial y} = 1 \qquad \dfrac{\partial^2 f}{\partial z^2} = 0$

$$\mathrm{Hess}_f(x, y, z) = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notation When we are dealing with derivatives of higher orders we use the following notation.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

let $m = (m_1, m_2 \ldots m_n)$

let $|m| := m_1 + m_2 + \cdots + m_n$.

For the partial derivative

$$\frac{\partial^{|m|} f}{\partial x_1^{m_1} \partial x_2^{m_2} \ldots \partial x_n^{m_n}} = \partial^{|m|}_{x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}} f = \boxed{\partial^{|m|}_{x^m} f}$$

x^m means $x_1^{m_1} . x_2^{m_2} \ldots x_n^{m_n}$.

We've seen a first order approximation to $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is given by.

$$f(x) = \underbrace{f(x_0) + \nabla f(x_0) \cdot (x - x_0)}_{\text{affine linear approx.}} + E(f, x, x_0)$$

Error $E(f, x, x_0)$ goes to 0 faster than $|x - x_0|$ does.

Ex Give an approximation for the number

$$\alpha = \sqrt{(3.03)^2 + (3.95)^2}$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto \sqrt{x^2+y^2}$$

$$\bar{x}_0 = (3,4) \qquad f(3,4) = 5$$

$$\nabla f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) \Bigg|_{\bar{x}_0 = (3,4)}$$

$$= \left(\frac{3}{5}, \frac{4}{5} \right)$$

$$f(3.03, 3.95)$$

$$\approx f(3,4) + \nabla f(3,4) \cdot \begin{pmatrix} 3.03-3 \\ 3.95-4 \end{pmatrix}$$

$$\approx 5 + \left(\frac{3}{5}, \frac{4}{5} \right) \cdot \begin{pmatrix} 0.03 \\ -0.05 \end{pmatrix}$$

$$\equiv 5 + \frac{3}{5} * 0.03 + \frac{4}{5} \cdot (-0.05)$$

$$\approx 4.978.$$

Actual value: $4.97829 \ldots$

For $f : \mathbb{R}^n \to \mathbb{R}$

An approximation to $f(y)$

for $y = (y_1 \cdots y_n)$ close to x_0

Defn

$$T_1 f(x_0; y) := f(x_0) + \nabla f(x_0) \cdot y$$

$$= f(x_0) + \frac{\partial f}{\partial x_1}(x_0) y_1 + \frac{\partial f}{\partial x_2}(x_0) y_2 + \dots + \frac{\partial f}{\partial x_n}(x_0) y_n.$$

is called the Taylor polynomial of f at point x_0 of order 1.

The second order Taylor poly at x_0 is

$$T_2 f(x_0; y) := f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2!} y \operatorname{Hess} f(x_0) \cdot y^t.$$

Ex: If $f : \mathbb{R}^2 \to \mathbb{R}$, $f \in C^2$

$a \in \mathbb{R}^2$, $y = y_1, y_2$.

$T_2 f(a;y)$

$$= f(a) + \frac{\partial f}{\partial x_1}(a) y_1 + \frac{\partial f}{\partial x_2}(a) y_2$$

$$+ \frac{1}{2!} \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= f(a) + \frac{\partial f}{\partial x_1}(a) y_1 + \frac{\partial f}{\partial x_2}(a) y_2$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(a) y_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) y_1 y_2$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(a) y_2^2.$$

We can generalize to higher dimensions

The Taylor poly of order k is

$$T_k f(x_0; y) := f(x_0)$$

$$+ \sum \frac{\partial f}{\partial x_i}(x_0) y_i$$

$$\cancel{+ \frac{1}{2!} \left(\frac{\partial^2 f}{\partial x_i^2} y_i^2 + 2 \frac{\partial^2 f}{\partial x_i \partial x_j} \right)}$$

$$\vdots$$

$$+ \sum_{m_1 + m_2 + \dots + m_n = k} \frac{1}{m_1! \, m_2! \cdots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) \cdot y_1^{m_1} \cdots y_n^{m_n}.$$

For ex:

degree 2 terms are

$$\sum_{m_1 + \dots + m_n = 2} \frac{1}{m_1! \cdots m_n!} \frac{\partial^2 f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) \, y_1^{m_1} \cdots y_n^{m_n}$$

With the multi-index notation we can write

that

$$\boxed{T_k f(x_0; y) = \sum_{|m| \leq k} \frac{1}{m!} \partial_x^{|m|} f(x_0)\, y^m}$$

$m! = m_1!\, m_2! \cdots m_n!$

$y^m = y_1^{m_1}\, y_2^{m_2} \cdots y_n^{m_n}$.

$$f(x) = T_k f(x_0, x - x_0) + E_k(f, x_0, x - x_0)$$

where

$$\lim_{x \to x_0} \frac{E_k(f, x, x - x_0)}{\|x - x_0\|^k} = 0.$$

<u>Thm</u> Taylor approximation.

Let $f \in C^k(X, \mathbb{R})$, $x_0 \in \overline{X}$

Then we have.

<u>Example</u>:

$$f(x,y) = e^{x+y}\cos x = e^x e^y \cos x$$

$$x_0 = (0,0)$$

$$T_3 f((0,0),(x,y)) = ?$$

$$f(0,0) = 1.$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left. e^x e^y \cos x - e^x e^y \sin x \right|_{(0,0)}$$

$$= 1.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \left. e^{x+y}\cos x \right|_{(0,0)} = 1$$

$$T_1 f\left((0,0)\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= 1 + 1\cdot x + 1\cdot y$$

$$1 + x + y.$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{0,0} = 0 \qquad \left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial y^2} = 1 \qquad \mathrm{Hess}_f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$(T_2 f)\left((0,0),\begin{pmatrix} x \\ y \end{pmatrix}\right) = 1 + x + y$$

$$+ \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + x + y + \frac{1}{2}0\cdot x^2 + xy + \frac{1}{2}y^2$$

$$T_2 f = 1 + x + y + xy + \frac{1}{2}y^2$$

$$\left.\frac{\partial^3 f}{\partial x^3}\right|_{(0,0)} = -2 \qquad \frac{\partial^3 f}{\partial y^3} = 1$$

$$\left.\frac{\partial^3 f}{\partial x^2 \partial y}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial^3 f}{\partial x \partial y^2}\right|_{(0,0)} = 1$$

$$T_3 f = 1 + x + y + xy + \frac{1}{2}y^2$$
$$+ \frac{1}{6}(-2)x^3 + \frac{1}{6}\cdot 1\, y^3$$
$$+ \frac{1}{1!\,2!}\, 1 \cdot x y^2$$

If you want approximate $f(0.1, 0.2)$ we can use

$$1 + 0.1 + 0.2 + (0.1)(0.2)$$
$$+ \frac{1}{2}(0.2)^2$$
$$+ \cdots \frac{1}{1!}\frac{1}{2!}\, 1 . (0.1)(0.2)^2$$

as an app. to $e^{0.3} \cdot \cos(0.1)$

§ 3.8 Critical points, extrema of functions

$$f: \mathbb{R}^n \to \mathbb{R}$$

Recall. For extrema of

$$f: \mathbb{R} \to \mathbb{R}$$

we looked at "critical points" ie points x_0 such that $f'(x_0)=0$.

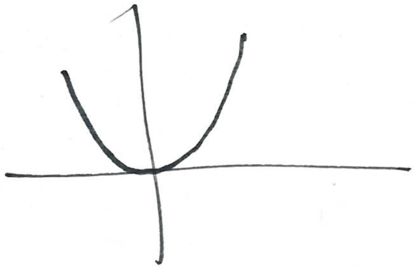
These are candidates for min or max of the function.

The nature of the extrema is determined by $f''(x_0)$

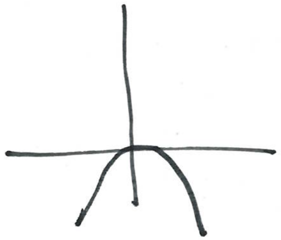
1) If $f''(x_0)>0$ then f is a local min

2) If $f''(x_0)<0$ then f is a local max.

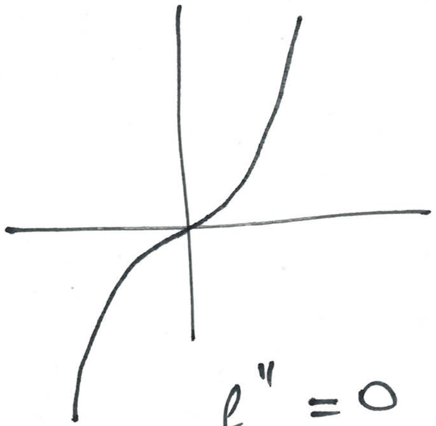
3) If $f''(x_0)=0$ it is a saddle pt.



$f'' > 0$



$f'' < 0$.



$f'' = 0$.

Defn $f : X \longrightarrow \mathbb{R}$

$X \subseteq \mathbb{R}^n$ open

A point $x_0 = (x_{0,1}, \dots, x_{0,n})$

Is a local max (resp. local min) if we can find a neighbourhood

$B_{x_0}(r) := \{ x \in \mathbb{R}^n \mid |x - x_0| < r \}$.

with $B_{x_0}(r) \subset X$ st

$\forall x \in B_{x_0}(r)$ $f(x) \leq f(x_0)$

(resp $(f(x) \geq f(x_0))$)

Thm $X \subset \mathbb{R}^n$ open

$f : X \to \mathbb{R}$ diff

if $x_0 \in X$ is a local

extrema (min or max of f)

then $\nabla f(x_0) = 0$

(This is the analog of

Thm: $f : \mathbb{R} \to \mathbb{R}$ diff

x_0 is an extreme of f

then $f'(x_0) = 0$.).

Defn A point $x_0 \in X$

is called *a critical point*

of f if $\nabla f(x_0) = 0$.