17-11-22

- $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$, $f \in C^2(X : \mathbb{R})$

$$\boxed{\mathrm{Hess}_f(x_0) := \left(\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}}$$

Hessian of f at x_0.

- $f : X \to \mathbb{R}$, $f \in C^2$

$$f(x) = \underbrace{\boxed{f(x_0) + \nabla f(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^t \mathrm{Hess}_f(x_0)(x - x_0)}}^{T_2 f(x - x_0, x_0)} + E_2(f, x, x_0)$$

- For $f \in C^k$

$$\boxed{T_k f(y; x_0) = \sum_{\substack{m \\ |m| \le k}} \frac{1}{m!} \partial_x^m f(x_0) y^m}$$

Taylor poly of f of order k at x_0.

$$\sum_{\substack{m \\ |m| = l}} \frac{1}{m!} \partial_x^m f(x_0) y^m = \sum_{m_1 + \cdots + m_n = l} (1/m_1! m_2! \ldots m_n!) \frac{\partial^l f(x_0)}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}} y_1^{m_1} \ldots y_n^{m_n}$$

Thm For $f \in C^k$, we have

$$f(x) = T_k f(x - x_0, x_0) + E_k(f, x, x_0)$$

where

$$\lim_{x \to x_0} \frac{E_k(f, x, x_0)}{\|x - x_0\|^k} = 0.$$

//

Extrema of $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$

Defn A point $x_0 \in X$ is a local maximum (resp min) of f if there is $r \in \mathbb{R}$, and $B_{x_0}(r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ with $B_{x_0}(r) \subset X$ such that $\forall x \in B_{x_0}(r)$
$f(x) \le f(x_0)$ (resp. $f(x) \ge f(x_0)$)

Thm Let $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$ differentiable.
If $x_0 \in X$ is a local extrema (i.e. local min or max) of f then $\nabla f(x_0) = 0$. [Analog of Thm: $f: \mathbb{R} \to \mathbb{R}$, x_0 is a local extrema, then $f'(x_0) = 0$].

Defn A point $x_0 \in X$ is called a critical point of f if $\nabla f(x_0) = 0$.

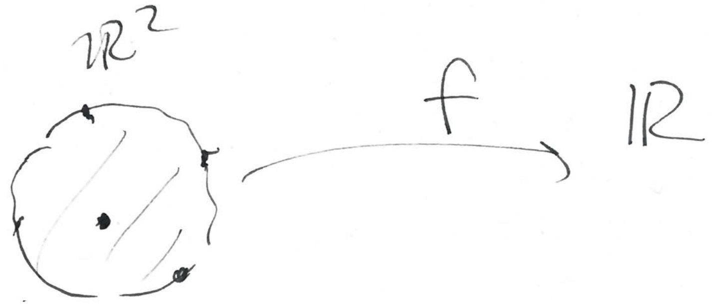
Critical points are candidates for local extrema.
A critical point which is not an extrema is called a saddle point

Recall : $f : [a,b] \to \mathbb{R}$.

differentiable in (a,b).

Then f attains its global min and max either at $x_0 \in (a,b)$ for which $f'(x_0) = 0$ or for $x_0 = a$ or $x_0 = b$.

Thm If $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$ is diff. on the interior of X, and if X is closed and bounded then ~~every~~ global extrema of f exists and is either at a point $x_0 \in$ interior of X for which $\nabla f(x_0) = 0$ or $x_0 \in$ boundary of X.



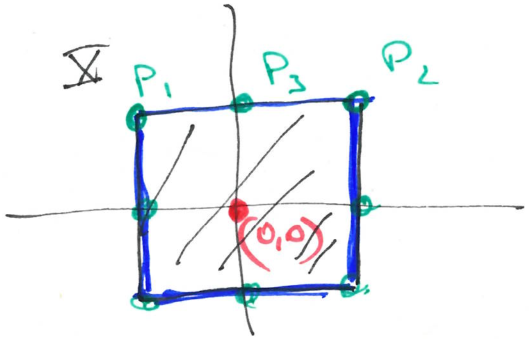
Rk. ~~For~~ ~~At~~ A diff. function on a closed bounded set extrema is either at

(1) x_0 w/ $\nabla f(x_0) = 0$.

(2) $x_0 \in$ boundary.

Example $f(x,y) = x^2+y^2$

let $X = [-1,1] \times [-1,1] \subset \mathbb{R}^2$



To find the extrema of f in X, (1) Find critical points inside X

$$\nabla f(x,y) = (2x, 2y) = (0,0)$$
$$\Rightarrow (x,y) = (0,0)$$

We also have to ~~find~~ check the boundary of X.

$(x,1)$ $(x,-1)$ horizontal boundries

$-1 \leq x \leq 1$

$(1,y)$ $(-1,y)$ vertical bound.

$-1 \leq y \leq 1$.

On the horizontal line $L_1 = \{(x,1) \mid -1 \leq x \leq 1\}$.

$f(x,y)\Big|_{L_1} = x^2+1$. we need to find extreme of x^2+1 for $-1 \leq x \leq 1$.

It has max at $x = \pm 1$
min at $x = 0$

$\max f\Big|_{L_1} = 2$ occurs at $P_1(-1,1), P_2(1,1)$

$\min f\Big|_{L_1} = 1$ occurs at $P_3(0,1)$

Similarly we see that
f takes value 2 at
$(-1,-1)$, $(1,-1)$
and value 1 at
$(0,-1)$, $(1,0)$, $(-1,0)$

To find the global
min and max of f
over the domain X
we check the values
of f at all these
points to see that
global max of f is 2.
occurs at $(\pm 1, \pm 1)$
global min of f is 0
occurs at $(0,0)$.

e)

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Question How do we determine the nature of a critical point x_0 of f

ie $\nabla f(x_0) = 0$.

For $f : \mathbb{R} \to \mathbb{R}$

we used the second derivative $f''(x_0)$ to determine the nature of the critical point.

Defn. A **non-degenerate critical point** x_0 is of $f \in C^2(X, \mathbb{R})$

if $\det(\mathrm{Hess}_f(x_0)) \neq 0$.

~~Note~~.

Recall $f : \mathbb{R} \to \mathbb{R}$

x_0 a critical point

$(f'(x_0) = 0)$.

$f''(x_0) > 0 \Downarrow$ x_0 is local min, $f''(x_0) = 0 \downarrow$ saddle point. $f''(x_0) < 0 \downarrow$ local max

A symmetric matrix

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$, $\det A \neq 0$

is called

① <u>Positive definite</u>

$\iff$ $x A x^t > 0 \quad \forall x \in \mathbb{R}^n$

$\iff$ all eigenvalues of A are positive.

$\iff$ all principal minors of A are positive

$$\begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & & \\ & & & \ddots \end{pmatrix} \qquad a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \vdots$$

we write $A > 0$.

② <u>negative definite</u>

$\iff$ $-A$ positive definite

$\iff$ all e-values are negative

(we write $A < 0$).

③ is <u>indefinite</u> if

A has positive and negative e-values.

<u>**Thm**</u> $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$

$f \in C^2(X, \mathbb{R})$

let $x_0 \in X$ be a critical point of f, $\nabla f(x_0) = 0$.

Then ① if $\mathrm{Hess}_f(x_0) > 0$

then x_0 is a loc min

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2) If $Hess_f(x_0) < 0$ then x_0 is a local max.

3) If $Hess_f(x_0)$ is indefinite then a saddle point.

Example. 1) $f(x,y) = x^2+y^2$

Only one critical point $x_0 = (0,0)$.

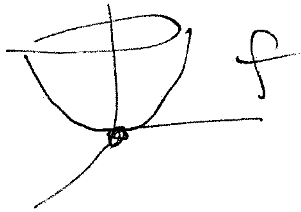
$$Hess_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0.$$

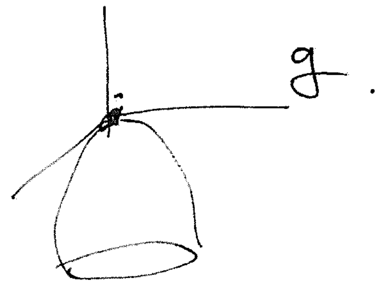
$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial^2 f}{\partial x^2} = 2.$$

$$\frac{\partial f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 2.$$

Hence $(0,0)$ is a local min.

2) $g(x,y) = -x^2-y^2$

1) 



Check $(0,0)$ is a local max.

3) $f(x,y) = xy$. $\nabla f = (y, x) = (0,0)$

$\Rightarrow y=0,\ x=0$.

$(0,0)$ is the only critical pt.

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

$$\frac{\partial f}{\partial x \partial y} = 1 \qquad \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 0.$$

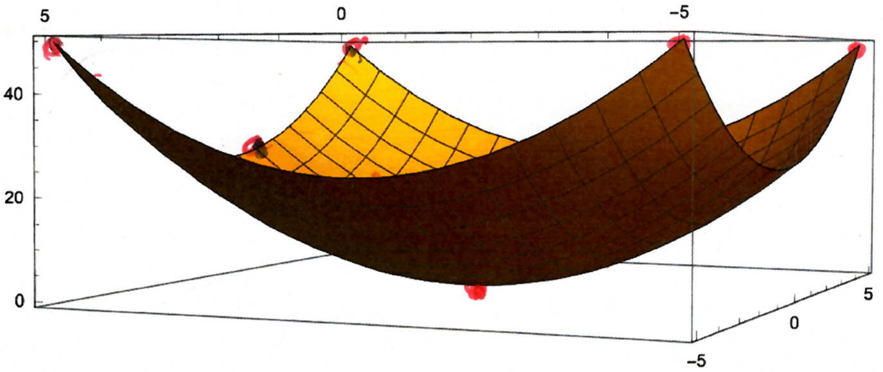
$$\mathrm{Hess}_f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Check that this has +ve and -ve eivalues.

indefinite $(0,0)$ is a saddle point.

In[1]:= `Plot3D[x^2 + y^2, {x, -5, 5}, {y, -5, 5}]`

Out[1]=



$f = [-5,5]\times[-5,5] \to \mathbb{R}$

$f(x,y) = x^2+y^2$

$g = [-5,5] \to \mathbb{R}$

$x \to f(x,5) = x^2+25$

$g'(x) = 2x = 0$

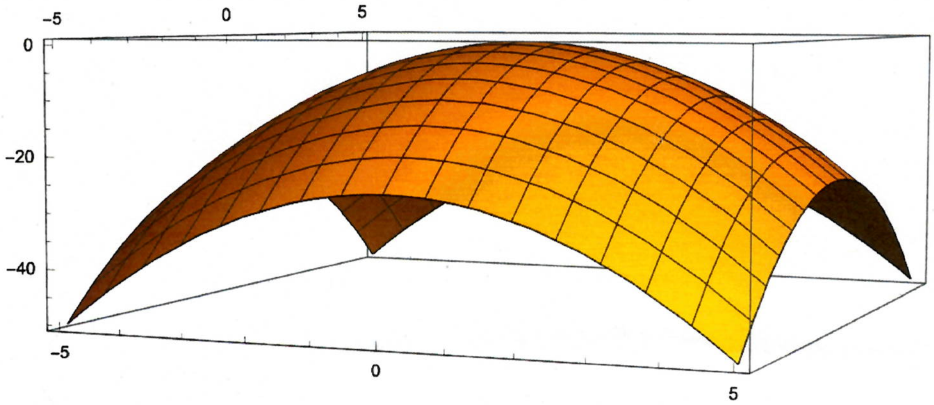
$\Rightarrow x=0.$

$\downarrow$

critical pt of $g(x)$

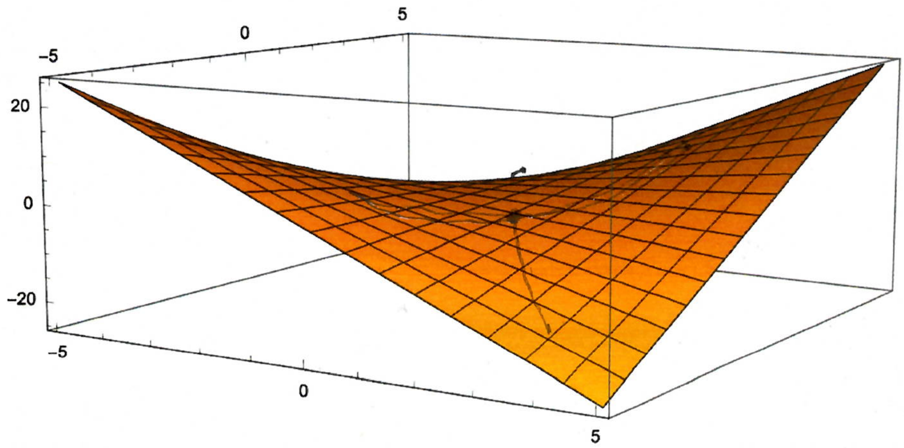
In[2]:= `Plot3D[-x^2 - y^2, {x, -5, 5}, {y, -5, 5}]`

Out[2]=



In[4]:= `Plot3D[x * y, {x, -5, 5}, {y, -5, 5}]`

Out[4]=



Ex: $f(x,y,z) = (x-1)^2 + (y+2)^2 + (z+1)^2$

$$\nabla f = \big(2(x-1),\ 2(y+2),\ 2(z+1)\big)$$

$$\nabla f = 0 \implies \begin{array}{l} x = 1 \\ y = -2 \\ z = -1 \end{array}$$

Only one critical pt.

$(1,-2,-1)$

$$\mathrm{Hess}_f(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} > 0.$$

Here $(1,-2,-1)$ is a loc. min.

Ex $f(x,y) = x \sin y$

$$\nabla f(x,y) = (\sin y,\ x\cos y) = (0,0)$$

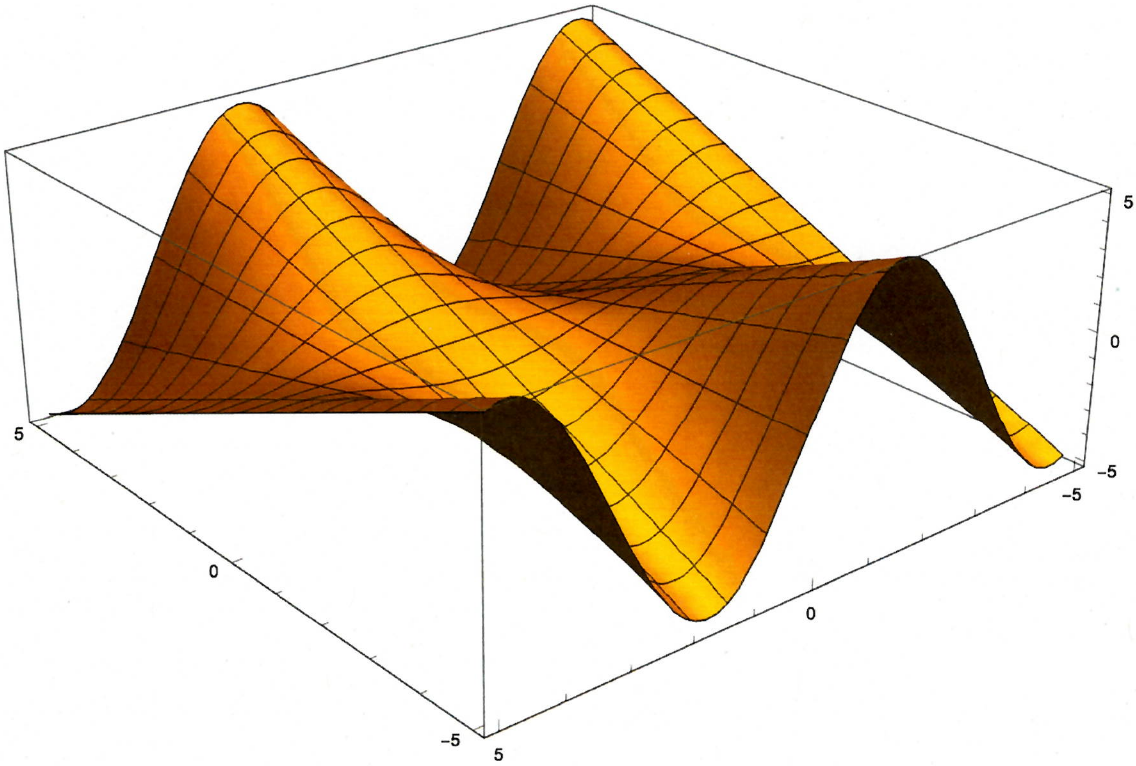
$$\left.\begin{array}{l} \sin y = 0 \\ x\cos y = 0 \end{array}\right\} \implies \begin{array}{l} y = \pi k. \\ x = 0. \end{array}$$

Critical points are

$(0, \pi k)$ $k \in \mathbb{Z}$.

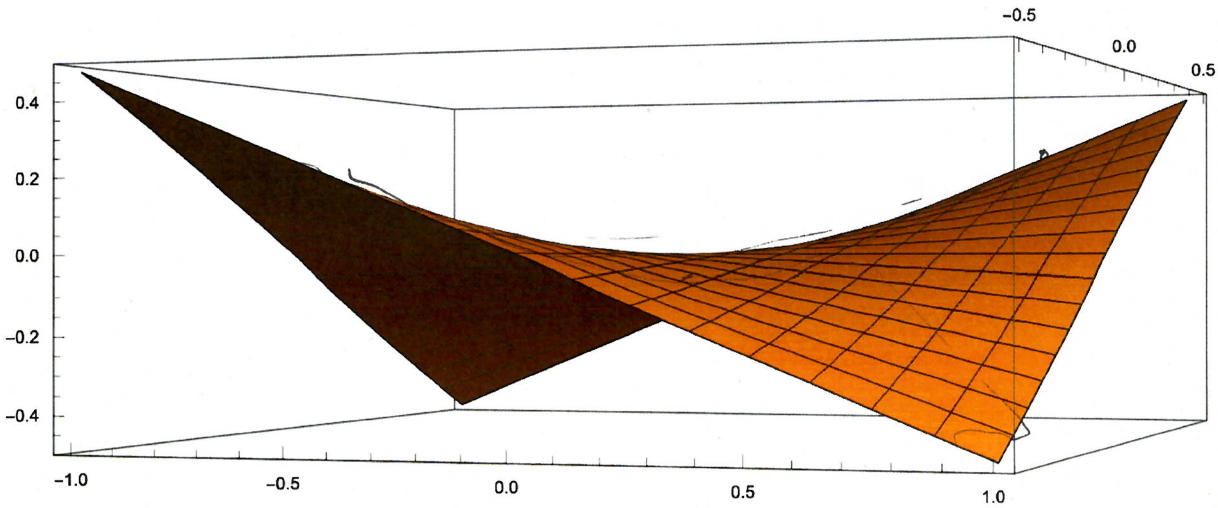
In[6]:= `Plot3D[x * Sin[y], {x, -5, 5}, {y, -5, 5}]`

Out[6]=



In[10]:= `Plot3D[x * Sin[y], {x, -1, 1}, {y, -0.5, 0.5}]`

Out[10]=



11 ½

$f_{xx} = 0$

$f_{xy} = \cos y$.

$f_{yy} = -x\sin y$.

$$\mathrm{Hess}_f(x,y) = \begin{pmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{pmatrix}.$$

$$\mathrm{Hess}_f(0, \pi k) = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

depending on whether k = even or odd.

Always indefinite w/ eivalues ± 1

critical points are saddle points

<u>Ex</u>: $f(x,y) = e^{\cos(x-y)} + x^2$

$$\nabla f = \left(e^{\cos(x-y)} \cdot -\sin(x-y) + 2x,\ e^{\cos(x-y)} \cdot \sin(x-y)\right).$$

$= (0,0)$

Since exp. is never zero

$$\frac{\partial f}{\partial y} = e^{\cos(x-y)} \sin(x-y) = 0$$

$\Rightarrow \sin(x-y) = 0.$

Putting this into $\frac{\partial f}{\partial x} = 0$ we get $2x = 0 \Rightarrow x = 0.$

if $x=0$ then $\sin(x-y) = \sin(-y) = 0$

gives $y = \pi k$

Critical points are $(0, \pi k)$ $k \in \mathbb{Z}$.

$$\frac{\partial^2 f}{\partial x^2} = 2 + e^{\cos(x-y)} \cdot (\sin(x-y))^2 - e^{\cos(x-y)} \cos(x-y)$$

@ $(0, \pi k)$

$$\frac{\partial^2 f}{\partial x^2}(0, \pi k) = 2 - e^{\pm 1}(\pm 1)$$

$$= \begin{cases} 2 - e & \text{if } k = \text{even} \\ 2 + e^{-1} & \text{if } k = \text{odd} \end{cases}$$

$$\frac{\partial^2 f}{\partial y^2} = e^{\cos(x-y)}(\sin(x-y))^2 + e^{\cos(x-y)}(-\cos(x-y))$$

$$\frac{\partial^2 f}{\partial y^2}(0, \pi k) = \mp e^{\pm 1}$$

$$= \begin{cases} -e & k \text{ even} \\ e^{-1} & k \text{ odd.} \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -e^{\cos(x-y)}(\sin(x-y))^2 + e^{\cos(x-y)} \cos(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, \pi k) = e^{\pm 1}$$

$$= \begin{cases} e & \text{if } k = \text{even.} \\ e^{-1} & \text{if } k = \text{odd} \end{cases}$$

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$$\mathrm{Hess}_f(0,\pi k) = \begin{cases} \begin{pmatrix} 2-e & e \\ e & -e \end{pmatrix} & \text{if } k = \text{even or } k=0. \\ \begin{pmatrix} 2+e^{-1} & -e^{-1} \\ -e^{-1} & e^{-1} \end{pmatrix} & \text{if } k=\text{odd}. \end{cases}$$

$\mathrm{Hess}_f(0,\pi k)$ is positive def. if $k=$ odd.

since $2+e^{-1}>0$

$\det = \frac{2}{e} + \frac{1}{e^2} - \frac{1}{e^2} = \frac{2}{e} > 0.$

if $k=$ even, Hess is indefinite since

$$-\mathrm{Hess}_f(0,\pi k) = \begin{pmatrix} e-2 & -e \\ -e & e \end{pmatrix}$$

$e-2>0$

but $\det = e^2 - 2e - e^2 = -2e < 0.$

Hence $(0,\pi k) = \begin{cases} \text{local min} & \text{if } k=\text{odd} \\ \text{saddle pt.} & \text{if } k=\text{even}. \end{cases}$

Question

What happens at a degenerate critical point?

ie $\nabla f(x_0)=0$, $\det \mathrm{Hess}_f(x_0)=0$

Then cannot conclude anything in general!!

eg. $f_1(x,y) = x^4+y^4$

$f_2(x,y) = -x^2-y^4$

$f_3(xy) = -x^4+y^4$

all have $\nabla f_i(0,0)=0$

$\mathrm{Hess}_{f_i}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$(0,0)$ is min pt for f_1, max for f_2

saddle pt for f_3.

§ Integration in $\mathbb{R}^n$.

Recall: $f : \mathbb{R} \to \mathbb{R}^n$
$x \to (f_1(x), \dots f_n(x))$

f is just a collection of n 1 variable functions

$f_i : \mathbb{R} \to \mathbb{R}$ $1 \leq i \leq n$.
$x \to f_i(x)$.

We've defined derivative of
$f'(x) = (f_1'(x), \dots f_n'(x))$

Similarly we define the integral of f from a to b as

$$\int_a^b f(x)\,dx = \begin{pmatrix} \int_a^b f_1(x)\,dx \\ \int_a^b f_2(x)\,dx \\ \vdots \\ \int_a^b f_n(x)\,dx \end{pmatrix}$$

In this chapter we'll either look at functions

(1) $f: \mathbb{R}^n \to \mathbb{R}^n$

Such functions are called vector fields.

and "integrate them along a curve".

(2) $f: \mathbb{R}^n \to \mathbb{R}$

and integrate them over regions in $\mathbb{R}^n$.

$f: \mathbb{R}^n \to \mathbb{R}^n$ are called vector fields because we can represent them in $\mathbb{R}^n$ by associating to every pt x_0 in $\mathbb{R}^n$ a vector $f(x_0)$ in $\mathbb{R}^n$

e.g $f: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (x^2+y^2, xy)$