24-11-22

Extrema of $f: X \to \mathbb{R}$

$X \subset \mathbb{R}^n$; differentiable

Thm If $x_0 \in X$ is a local extrema of f then

$\nabla f(x_0) = 0$.

Thm If $f: X \to \mathbb{R}$ is differentiable in the interior of X, and if X is closed and bounded, then global extrema of f exists and it is either at a point x_0 with

$\nabla f(x_0) = 0$ or at a point

on the boundary of X.

x_0 is a non-degenerate critical point of $f \in C^2(X: \mathbb{R})$ if $\det(\mathrm{Hess}_f(x_0)) \neq 0$

Thm $f: X \to \mathbb{R}$, $f \in C^2$.

$x_0 \in X$ a critical point of f

Then ① If $\mathrm{Hess}_f(x_0) > 0$

then x_0 is a local minimum

② If $\mathrm{Hess}_f(x_0) < 0$ then

x_0 is a local maximum

③ If $\mathrm{Hess}_f(x_0)$ is indefinite

then x_0 is a saddle point.

Rk If $X \subset \mathbb{R}^2$ $f: X \to \mathbb{R}$

$\mathrm{Hess}_f(x_0) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ then

1) x_0 is a loc. min if $a > 0$, $ac - b^2 > 0$
2) x_0 is a loc. max if $a < 0$, $ac - b^2 > 0$
3) x_0 is a saddle pt if $ac - b^2 < 0$

Line Integrals

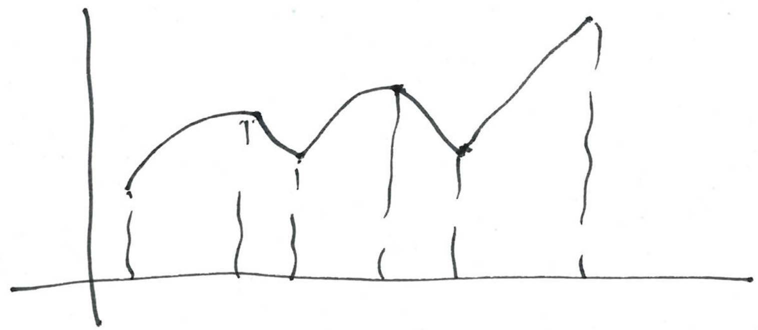
(Path Integrals)

let γ be vector valued function on an interval $[a, b]$

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n$$
$$t \longmapsto (\gamma_1(t), \gamma_2(t) \ldots \gamma_n(t))$$

which is continuous and piecewise C^1, ie $\exists$ $k > 1$ and partition $a = t_0 < t_1 < \ldots < t_k = b$

$$\gamma\Big|_{[t_{k-1}, t_k]} \in C^1$$



We say γ is a parametrized curve, and $\gamma(t)$ is a parametrization of the curve

$$\operatorname{Im} \gamma \doteq \gamma([a, b])$$

Examples

1) $\gamma : [0,1] \to \mathbb{R}^3$

$t \mapsto (1+t,\ 2+t,\ 3-t)$

is the parametrization of the segment in $\mathbb{R}^3$ which goes through the point $\vec{a} = (1,2,3)$ in the direction of $\vec{b} = (1,1,-1)$

at $t=1$ you are at $(2,3,2) = a+b = c$

$\gamma(t) = a(1-t) + ct$

If $t \in \mathbb{R}$ then we get the line

$r(t) = \vec{a} + \vec{b}t$.

2) In general if $\vec{x}_0, \vec{x}_1$ are 2 points in $\mathbb{R}^n$

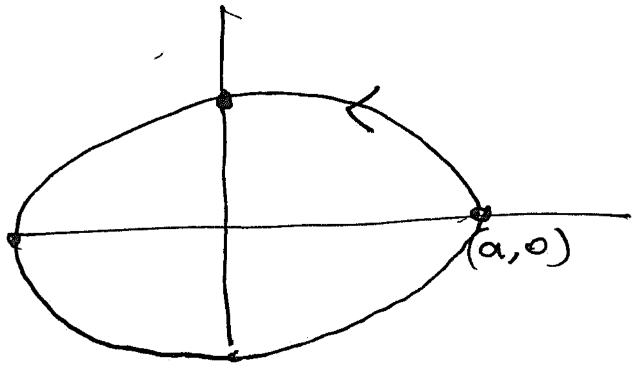
Then $\gamma : [0,1] \to \mathbb{R}^n$

$t \to \vec{x}_0(1-t) + \vec{x}_1 t$

Is the line segment which starts at $\vec{x}_0$ and ends at $\vec{x}_1$.

③ $\gamma : [0, 2\pi] \to (a\cos t, b\sin t)$

$\quad\quad\quad\ t$



Is a parametrization of an Ellipse.

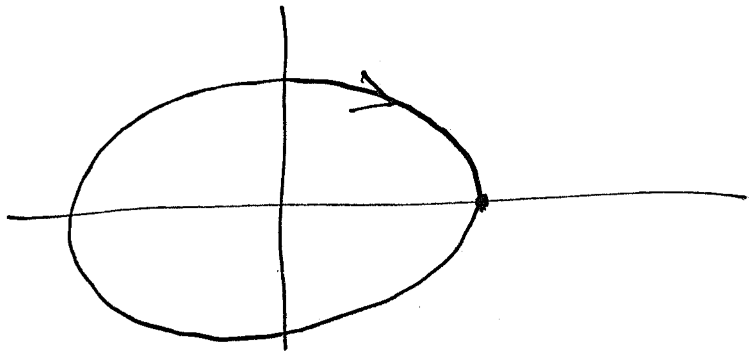
If $a=b$ then

$\gamma : [0, 2\pi] \to (a\cos t, a\sin t)$

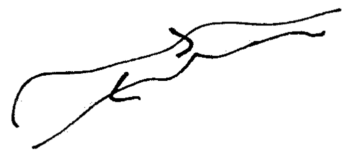
Is a parametrization of the circle of radius a traced in counterclockwise direction.

④ $\alpha : [0, 2\pi] \to \mathbb{R}^2$

$t \to (a\cos(2\pi - t), b\sin(2\pi - t))$



traces the ellipse in the opposite direction



⑤. In general

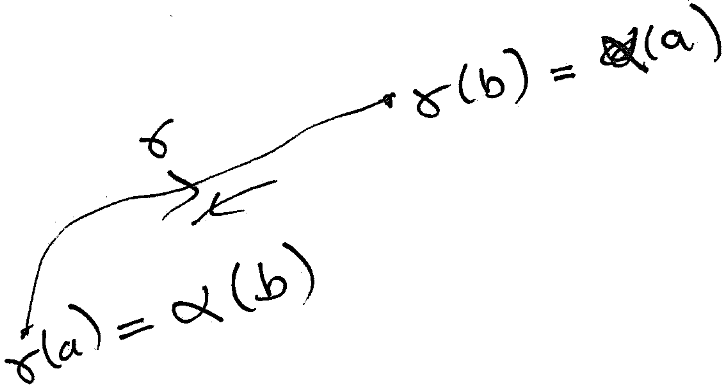
if $\gamma : [a,b] \to \mathbb{R}^n$
$t \to \gamma(t)$

is a curve

then $\alpha : [a,b] \to \mathbb{R}^n$

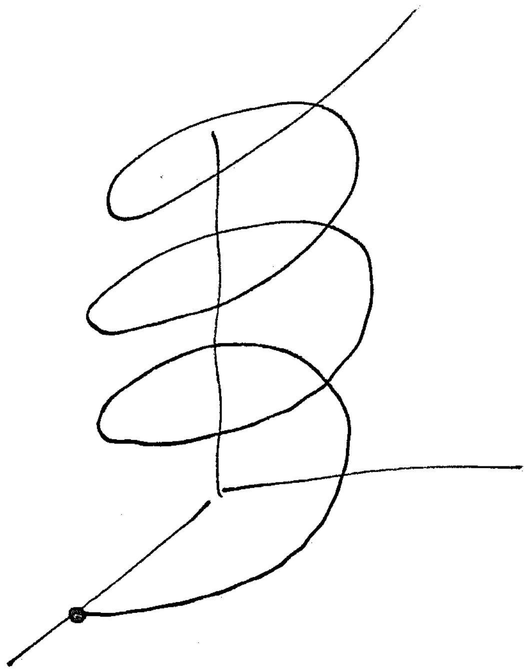
$\alpha(t) := \gamma(b+a-t)$

traces the same curve in the opposite direction



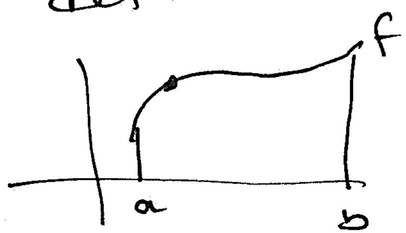
⑥ $\gamma : \mathbb{R} \to \mathbb{R}^3$

$t \to (a\cos t, a\sin t, t)$



⑦ If $f : \mathbb{R} \to \mathbb{R}$ a function. Then its graph determines a curve in $\mathbb{R}^2$

$\gamma : [a,b] \to \mathbb{R}^2$
$t \to (t, f(t))$

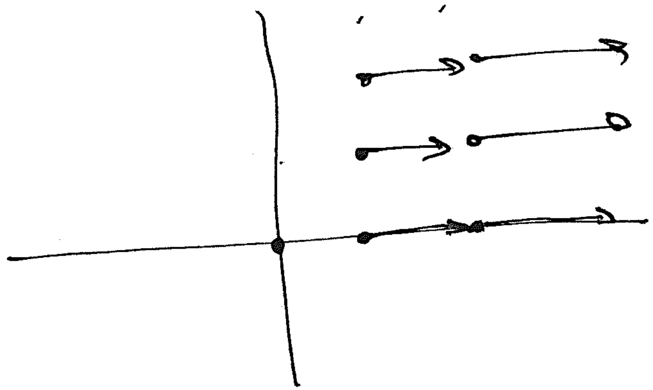


let $v : \mathbb{R}^n \to \mathbb{R}^n$

be a vector field.

eg. $v : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (x,0)$



Defn. let $\gamma : [a,b] \to \mathbb{R}^n$

a curve in $\mathbb{R}^n$. $X \subset \mathbb{R}^n$

a subset of $\mathbb{R}^n$ which

contains the image of γ.

$v : X \to \mathbb{R}^n$ a

continuous function

The Integral

$$\int_a^b \underbrace{\Big(\overbrace{v(\gamma(t))}^{\in \mathbb{R}^n} \underset{\substack{\downarrow \\ \text{scalar product}}}{\cdot} \overbrace{\gamma'(t)}^{\in \mathbb{R}^n} \Big)}_{\in \mathbb{R}} \, dt$$

is called the line or the path integral of v along γ.

Other Notations

$$\int_\gamma v \cdot ds$$

If $v = (v_1(x), v_2(x), \ldots v_n(x))$

$\gamma(t) = (\gamma_1(t), \gamma_2(t) \ldots \gamma_n(t))$

ds represents $\gamma'(t)dt$

If $n=2$ $\quad \gamma : I \to \mathbb{R}^2$

$\quad t \to (\gamma_1(t), \gamma_2(t))$

$$\int_\gamma v\, ds = \int_a^b \begin{pmatrix} v_1(\gamma(t)) \\ v_2(\gamma(t)) \end{pmatrix} \cdot \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$$

$$\int_a^b \left[v_1(\gamma(t)) \cdot \gamma_1'(t) + v_2(\gamma(t))\, \gamma_2'(t) \right] dt$$

$$\int_a^b v_1(x,y)\,dx + v_2(x,y)\,dy$$

$x = \gamma_1(t) \qquad y = \gamma_2(t)$

$dx = \gamma_1'(t)dt \qquad dy = \gamma_2'(t)dt$

__Ex__ 1 $v(x,y) = (-y, x)$

$\gamma(t) = (\cos t, \sin t)$

$0 \leq t \leq 2\pi$

$$\int_0^{2\pi} \langle v(\gamma(t)), \gamma'(t) \rangle\, dt$$

$$v(\gamma(t)) = v(\cos t, \sin t)$$
$$= (-\sin t, \cos t)$$

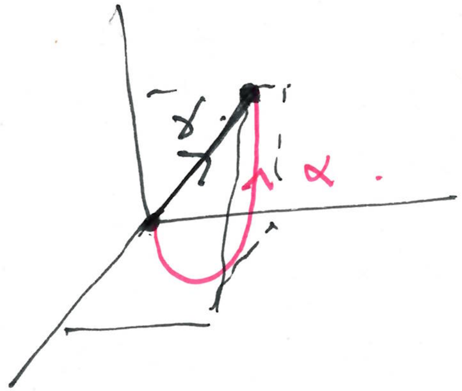
$$\gamma'(t) = (-\sin t, \cos t)$$

$$\langle v(\gamma(t)), \gamma'(t)\rangle$$
$$\sin^2 t + \cos^2 t = 1$$

$$\int v\,ds = \int_0^{2\pi} 1\,dt = 2\pi$$

② . $f: \mathbb{R}^3 \to \mathbb{R}^3$
$$(x,y,z) \to (y^2, xz, 1)$$

$$\gamma: [0,1] \to \mathbb{R}^3$$
$$t \to (t,t,t)$$
$$\gamma'(t) = (1,1,1)$$



$$\alpha: [0,1] \to \mathbb{R}^3$$
$$t \to (t, t^2, t^3)$$
$$\alpha'(t) = (1, 2t, 3t^2)$$

$$\int_\gamma f\,ds = \int_0^1 \langle f(\gamma(t)), \gamma'(t)\rangle\,dt$$

$$= \int_0^1 \langle (t^2, t^2, 1), (1,1,1)\rangle\,dt$$

$$= \int_0^1 2t^2+1\,dt = \frac{2t^3}{3} + t \Big|_0^1 = 5/3$$

/8.

$$\int_\alpha f\,ds = \int_0^1 \langle f(\alpha(t)), \alpha'(t)\rangle\, dt$$

$$\int_0^1 \langle f(t, t^2, t^3), (1, 2t, 3t^2)\rangle\, dt$$

$$\int_0^1 (t^4, t^4, 1)\cdot(1, 2t, 3t^2)\, dt$$

$$= \int_0^1 t^4 + 2t^5 + 3t^2\, dt$$

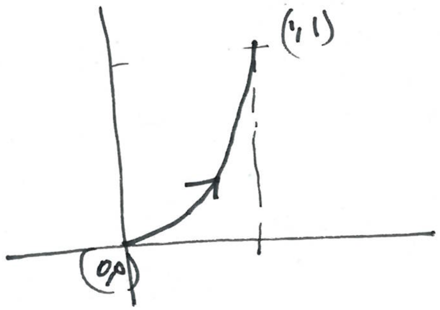
$$= \dots$$

$$= 23/15.$$

Clicker

$$V : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \mapsto (x^2 - 2y,\ 2x + y)$$



$$\gamma : [0,1] \to \mathbb{R}^2$$

$$t \to (t, t^3)$$

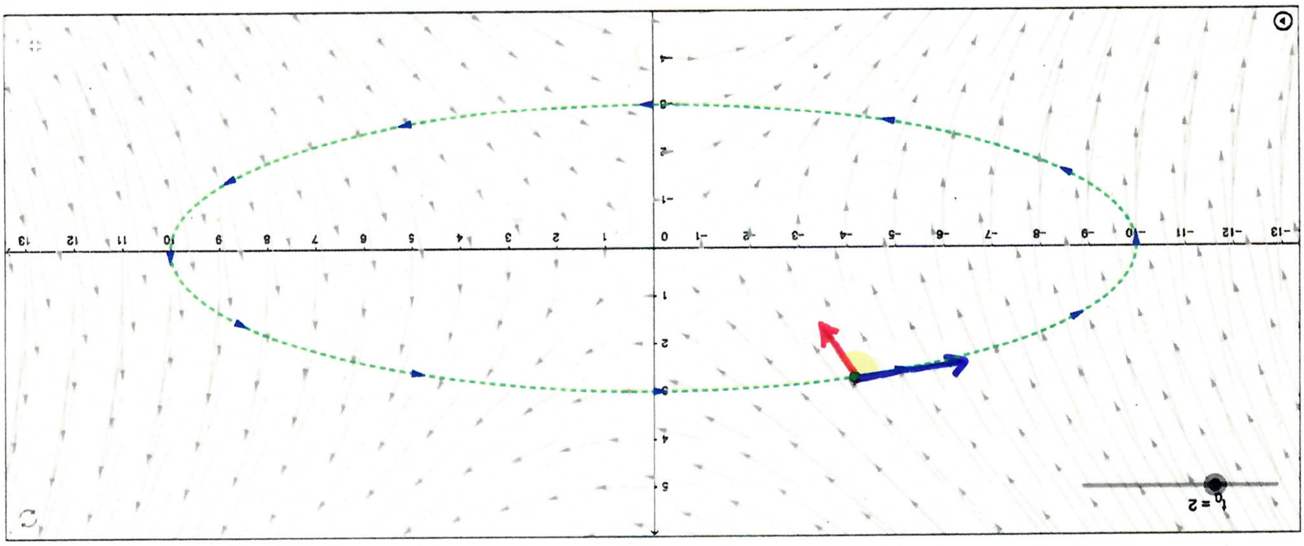
$$v(\gamma(t)) = v(t, t^3) = (t^2 - 2t^3,\ 2t + t^3)$$

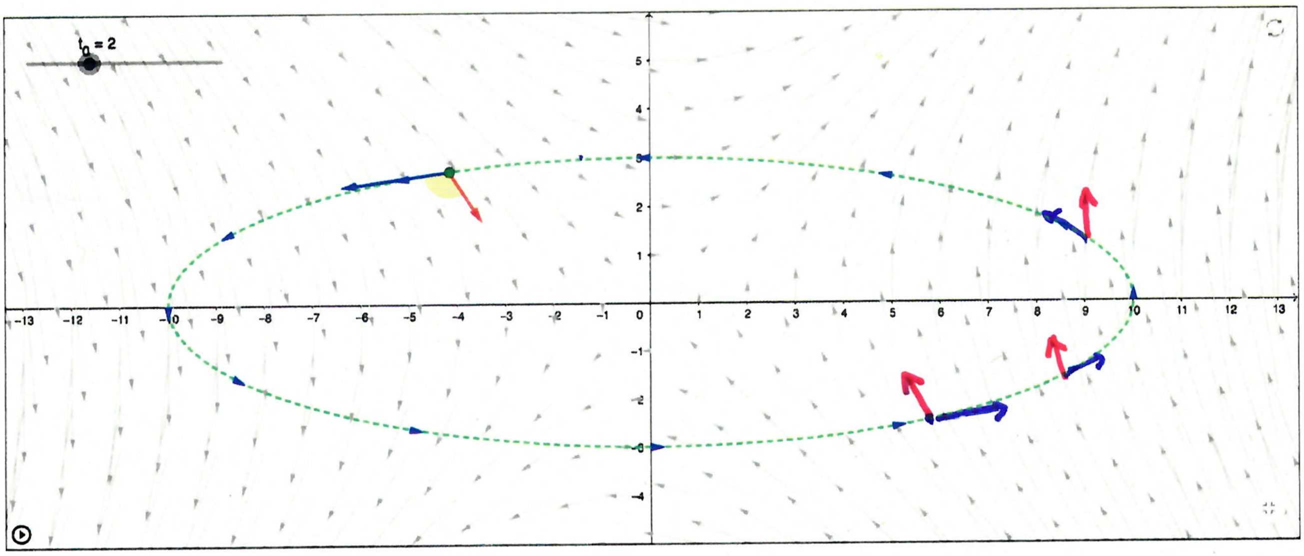
$$\gamma'(t) = (1, 3t^2)\, dt$$

$$\int_0^1 \langle V(\gamma(t)), \gamma'(t)\rangle\, dt = \int_0^1 t^2 - 2t^3 + 6t^3 + 3t^5$$

$$= \int_0^1 t^2 + 4t^3 + 3t^5\, dt$$

$$= \dots 11/6.$$





scalar product

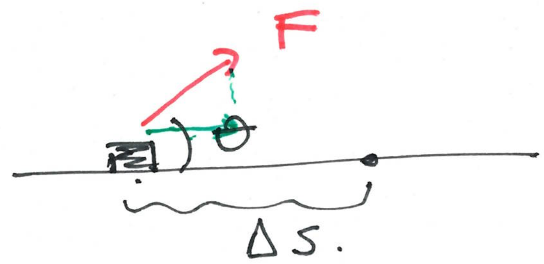
$$\int_a^b f(s)\,ds := \int_a^b \underbrace{f(\gamma(t))}_{\substack{\downarrow \\ \text{red vector}}} \cdot \underbrace{\gamma'(t)}_{\substack{\downarrow \\ \text{blue vector}}} \, dt$$

Geogebra: Line Integral of a vector field in 2space

Author: Kristen Beck

https://www.geogebra.org/m/YZWyyM47

Why do we define such an integral?



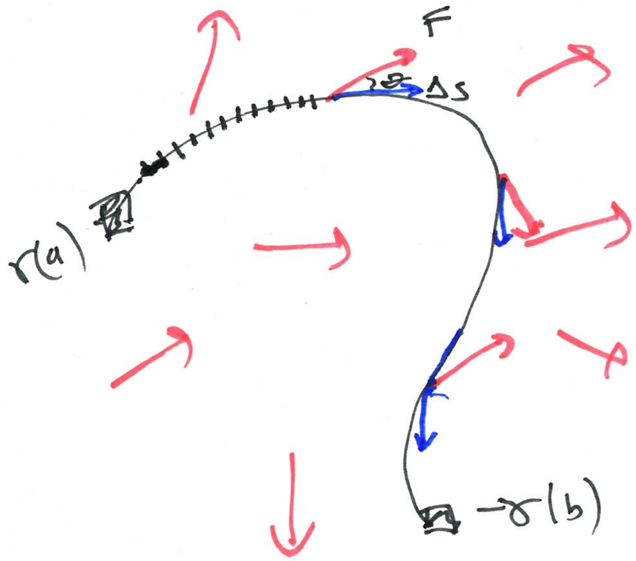
Assume you have a point mass that moves under influence of a constant vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$ along a line.

If it is moved by a distance Δs, then the work done is given by

$$W = \vec{F} \cdot \vec{\Delta s} = |\vec{F}|\,|\vec{\Delta s}| \cdot \cos\theta$$

$$\underbrace{|F| \cos\theta}_{x \longrightarrow y}$$

Now suppose it is moved along a curve under the influence of a force field that changes at every pt.



$$\Delta w = \vec{F} \cdot \Delta s$$

Divide the curve into small pieces $\Delta \gamma_i = \gamma_i(t_{i+1}) - \gamma(t_i)$

$$W = \sum \Delta w_i = \sum F(\gamma(t_i)) \cdot \Delta \gamma_i$$

$$\sum F(x(t_i), y(t_i)) \cdot \frac{\Delta \gamma_i}{\Delta t} \cdot \Delta t$$

let $\Delta t \to 0$.

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

Properties of the path integrals

1) It is independent of orientation preserving reparametrization of the curve.

reparametrization: if $\gamma : [a,b] \to \mathbb{R}^n$ $\sigma : [c,d] \to [a,b]$ which is C^1 s.t $\sigma(c) = a$, $\sigma(d) = b$ and $\sigma'(t) > 0$ then $\tilde{\gamma} := \gamma \circ \sigma : [c,d] \to \mathbb{R}^n$.

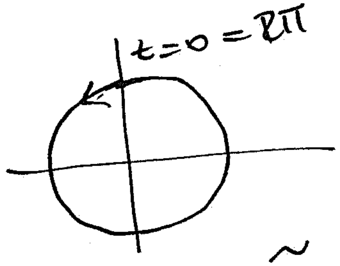
is a reparametrization of γ.

For the line integral:

$f : \mathbb{R}^n \to \mathbb{R}^n$
a vector field

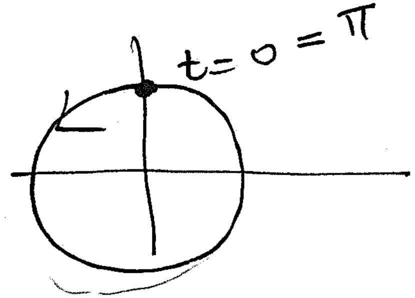
$$\boxed{\int_{\gamma} f \, ds = \int_{\tilde{\gamma}} f \, ds}$$

<u>Ex</u>: $\gamma : (0, 2\pi] \to \mathbb{R}^2$
$t \to (\sin t, \cos t)$



$\sigma = [0, \pi] \to [0, 2\pi]$
$t \to 2t$

$\tilde{\gamma} = \gamma \circ \sigma : [0, \pi] \to \mathbb{R}^2$
$t \to (\sin 2t, \cos 2t)$



$$\int_c^d f((\gamma \circ \sigma)(t)) \cdot (\gamma \circ \sigma)'(t) \, dt$$

$$= \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt.$$

$$\int_c^d f(\gamma(\sigma(t))) \cdot \gamma'(\sigma(t)) \cdot \sigma'(t) \, dt.$$

Let $\sigma(t) = u$. $\sigma'(t)\cancel{dt} = du$.

$$\int_a^b f(\gamma(u)) \, \gamma'(u) \cdot du$$

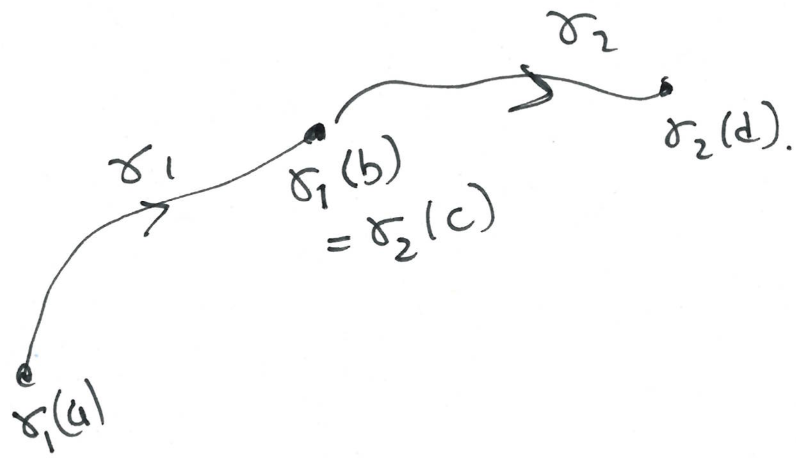
(2) Let $\gamma_1 : [a,b] \to X \subset \mathbb{R}^n$.

$\gamma_2 : [c,d] \to X$

2 paths with $\gamma_1(b) = \gamma_2(c)$

Define $\gamma_1 + \gamma_2$ as the path formed by concatenation of these 2 curves.

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a,b] \\ \gamma_2(t-b+c) & t \in [b, d+b-c]. \end{cases}$$



Then $\displaystyle\int_{\gamma_1+\gamma_2} f\,ds = \int_{\gamma_1} f\,ds + \int_{\gamma_2} f\,ds$.

③. If $\gamma : [a,b] \to \mathbb{R}^n$ a path. Let $-\gamma : [a,b] \to \mathbb{R}^n$ same path traced in the opposite direction

$$(-\gamma)(t) := \gamma(a+b-t)$$

Then $\displaystyle\int_{-\gamma} f\,ds = -\int_{\gamma} f\,ds.$

Analog of $\displaystyle\int_{[a,b]} f\,dx = \int_a^b f\,dx$

$$= -\int_b^a f\,dx$$

Recall fund. thm.

If $F' = f$

$$\int_a^b f\,dx = \int_a^b F'(x)\,dx$$

$$= F(b) - F(a).$$

Is there an analog for the path integral?

Important example.

Suppose $f : X \to \mathbb{R}^n$

$X \subset \mathbb{R}^n$ a vector field

such that $\exists\, g : X \to \mathbb{R}$

$g \in C^1$ so that $f = \nabla g$.

Suppose $\gamma : [a, b] \to X$

$$\int_\gamma f \, ds = \int_\gamma \nabla g \, ds$$

$$= \int_a^b \underbrace{\nabla g(\gamma(t)) \cdot \gamma'(t)} \, dt$$

By chain rule

$$\frac{d}{dt}(g \circ \gamma) = \nabla g(\gamma(t)) \cdot \gamma'(t)$$

$$\int_\gamma f \, ds = \int_\gamma \nabla g \, ds$$

$$= \int_a^b \left(\frac{d}{dt}(g \circ \gamma)\right) dt$$

$$= (g \circ \gamma)(b) - (g \circ \gamma)(a)$$

$$= g(\gamma(b)) - g(\gamma(a))$$

Rk Integral of f only depends on the end points of γ and of course on g.

Defn A differentiable function $g: X \subset \mathbb{R}^n \to \mathbb{R}$ so that $\nabla g = f$ is called a potential for f

Rk (1) $n=1$ a potential is simply a primitive.

(2) If $f: \mathbb{R} \to \mathbb{R}$ and continuous, then

$$g(x) := \int_a^x f(t)\,dt$$

is a primitive of f ie $g'(x) = f(x)$.

So for $f: \mathbb{R} \to \mathbb{R}$ continuous, primitive always exists.

Question for $f: \mathbb{R}^n \to \mathbb{R}^n$

Is there always a g such that $\nabla g = f$?

Answer: NO!

Ex.: $f(x,y) = (2xy^2, 2x)$

Suppose there is a g s.t $\nabla g = (2xy^2, 2x)$

$$\frac{\partial g}{\partial x} = 2xy^2 \qquad \frac{\partial g}{\partial y} = 2x$$

$$\Downarrow$$

$$g(x,y) = x^2y^2 + h(y)$$

$$\frac{\partial g}{\partial y} = 2x^2y + h'(y)$$

$$= 2x$$

Has no soln.

Hence no such g exists.

Ex: $f(x,y) = (2xy^2, 2yx^2)$

Is there a $g : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\frac{\partial g}{\partial x} = 2xy^2 \qquad \frac{\partial g}{\partial y} = 2yx^2$$

$$\Downarrow$$

$$g(x,y) = x^2y^2 + h(y)$$

$$\frac{\partial g}{\partial y} = 2yx^2 + h'(y)$$

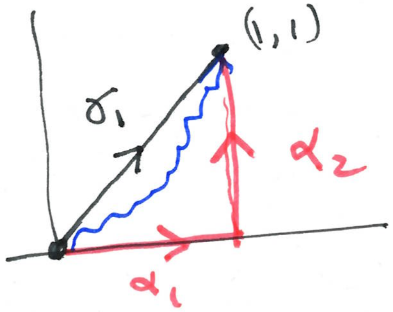
$$= 2yx^2$$

$$\Rightarrow h'(y) = 0 \Rightarrow h = \text{const.}$$

$$g(x,y) = x^2y^2 + c.$$

then $\nabla g = f$.

Ex: $f = (2xy^2, 2yx^2)$



$$\gamma_1 : [0,1] \to \mathbb{R}^2$$
$$t \to (t,t)$$

$$\alpha_1 : [0,1] \to \mathbb{R}^2$$
$$t \to (t,0)$$

$$\alpha_2 : [0,1] \to \mathbb{R}^2$$
$$t \to (0,t)$$

$$\int_\gamma f\,ds = 1 = \int_{\alpha_1+\alpha_2} f\,ds = 1$$

The integral is indep. of the path from (0,0) to (1,1)

Defn Let $X \subset \mathbb{R}^n$, $f : X \to \mathbb{R}^n$ be a continuous vector field. If for any $x_1, x_2 \in X$ the line integral

$\int_\gamma f\,ds$ of a curve in X from x_1 to x_2

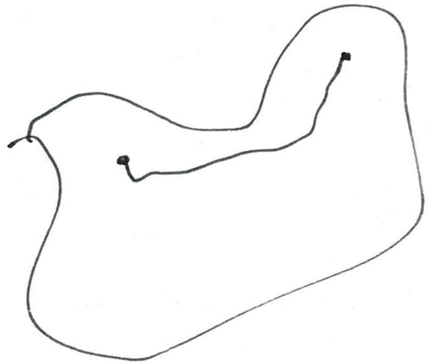
is independent of the curve then f is called conservative

How do we decide if a vector field is conservative?

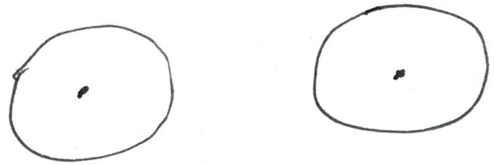
Defn. Let $X \subset \mathbb{R}^n$ open

X is path connected if for every pair of points in X, $x, y \in X$, $\exists$ a C^1 path $\gamma : [0,1] \to X$ with

$$\gamma(0) = x$$
$$\gamma(1) = y$$
$$\gamma([0,1]) \subset X$$



path connected



not path connected.

Thm: f continuous v. field on an open path connected set $X \subset \mathbb{R}^n$. Then FAE.

① f is the gradient of a function $g : X \to \mathbb{R}$

ie $f = \nabla g$

2) ~~[illegible]~~ line integral of f is independent of the path between the 2 points

3) The line integral of f along closed paths are zero.

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