Defn

1-11-22

$f : \mathbb{R}^n \to \mathbb{R}^n$ a vector field

$\gamma : [a,b] \to \mathbb{R}^n$ a curve

The line (path) integral of f along γ is

$$\boxed{\int_\gamma f \cdot ds := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt}$$

Properties of the path integral

① It is independent of orientation preserving reparametrization of the path γ. ie. if $\tilde{\gamma} : [c,d] \to \mathbb{R}^n$ is another parametrization;

$\tilde{\gamma}(t) := \gamma(\sigma(t))$ where

$\sigma : [c,d] \to [a,b]$ is $C^1([c,d])$

with $\sigma(c) = a$, $\sigma(d) = b$ and

$\sigma'(t) > 0$ $\forall\ t \in [c,d]$, then

$$\boxed{\int_\gamma f\, ds = \int_{\tilde{\gamma}} f \cdot ds}$$

② If $\gamma_1 : [a,b] \to \mathbb{R}^n$

$\gamma_2 : [c,d] \to \mathbb{R}^n$

2 curves and

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a,b] \\ \gamma_2(t-b+c) & t \in [b, d+b-c] \end{cases}$$

is the path formed by concatenation of γ_1 and γ_2

Then
$$\boxed{\int_{\gamma_1+\gamma_2} f\, ds = \int_{\gamma_1} f \cdot ds + \int_{\gamma_2} f\, ds.}$$

③ If $\gamma : [a,b] \to \mathbb{R}^n$ is a path and $-\gamma$ the same path traced in the opposite direction, ie

$(-\gamma)(t) := \gamma(a+b-t)$, then

$$\boxed{\int_{-\gamma} f\, ds = -\int_\gamma f \cdot ds.}$$

Important example

Suppose $f : X \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$

a vector field such that

$\exists\ g : X \to \mathbb{R}$, $g \in C^1[X : \mathbb{R}]$

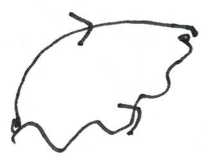
so that $f = \nabla g$.

let $\gamma : [a, b] \to X$ a curve in X

Then $$\int_\gamma f\, ds = \int_\gamma \nabla g . ds$$

$$= g(\gamma(b)) - g(\gamma(a))$$

ie the path integral of f depends only on the values of g at the end points of the curve; $\gamma(b)$ and $\gamma(a)$.



Remarks ① Not every $f : \mathbb{R}^n \to \mathbb{R}^n$ has such a g!

eg: $f(x,y) = (2xy^2, 2x)$

then $\exists$ no $g : \mathbb{R}^2 \to \mathbb{R}$ s.t

$f = \nabla g$.

② By the important example

if $\exists\ g$ s.t $f = \nabla g$ then

$\int_\gamma f\, ds$ only depends on the end points of γ

and not on the path γ.

Defn $f : X \to \mathbb{R}^n$ continuous V. field.

If for any $x_1, x_2 \in X$ the line integrals $\int_\gamma f . ds$ for any curve γ between x_1, x_2 are equal, (ie the line integral is indep. of the path) f is called conservative

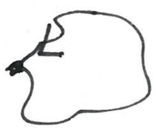
Thm let $f: X \to \mathbb{R}^n$ continuous vector field, X open and path connected sset of $\mathbb{R}^n$. Then TFAE.

1. f is the gradient of a function $g: X \to \mathbb{R}$ ie $f = \nabla g$

2. The line integral of f is independent of the path between 2 points.

3. The line integral of f along any closed path is zero

$\gamma: [a, b] \to X$ is closed when $\gamma(a) = \gamma(b)$



Defn when $f: X \to \mathbb{R}^n$ is the gradient of $g: \mathbb{R}^n \to \mathbb{R}$ for some g then g is called a **potential** for f

Q: Is there an easy criteria to check if a vector field $f: X \to \mathbb{R}^n$ is conservative?

Ans: We have a necessary condition!

Thm let $X \subseteq \mathbb{R}^n$ open

$f: X \to \mathbb{R}^n$ C^1 vektor field

$f(x) = (f_1(x), \ldots, f_n(x))$

If f is conservative then

$$\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j} \qquad 1 \le i, j \le n.$$

Pf If $f = \nabla g = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots \frac{\partial g}{\partial x_n}\right)$

then $g \in C^2$

Then mixed partials of 2nd order are equal.

$$\frac{\partial}{\partial x_i}\left(\frac{\partial g}{\partial x_j}\right) = \frac{\partial}{\partial j}\left(\frac{\partial g}{\partial x_i}\right)$$

Hence $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (\underset{f_1}{2xy^2}, \underset{f_2}{2x})$

$$\frac{\partial f_1}{\partial \cancel{x}y} = 4xy \qquad \frac{\partial f_2}{\partial x} = 2$$

$\neq$

$\Rightarrow f$ is not conservative

14.

$\overline{x} = (x, y, z)$

Rk. $n = 3$ $\quad f : \mathbb{R}^3 \to \mathbb{R}^3$

$(x,y,z) \to f_1(\overline{x}), f_2(\overline{x}), f_3(\overline{x})$

$$\frac{\partial f_3}{\partial x} = \frac{\partial f_1}{\partial z}$$

$$\frac{\partial f_3}{\partial y} = \frac{\partial f_2}{\partial z}$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$$

$$\partial_y f_3 - \partial_z f_2 = 0$$

$$\partial_z f_1 - \partial_x f_3 = 0$$

$$\partial_x f_2 - \partial_y f_1 = 0.$$

Defn Let $X \subset \mathbb{R}^3$

$f : X \to \mathbb{R}^3$ C^1 vector field.

Then the curl of f is defined

$$\boxed{\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}}$$

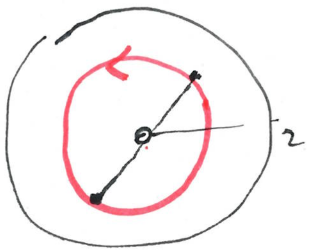
Thm $f : \mathbb{R}^3 \to \mathbb{R}^3$. f is cons $\Rightarrow$ curl $f = 0$

The defn of curl can be remembered as the formal determinant

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

<u>Ex</u> $X = \{(x,y) \mid 0 < x^2+y^2 < 2\}$

$$f(x,y) = \begin{pmatrix} -y/x^2+y^2 \\ x/x^2+y^2 \end{pmatrix}$$



$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$ check!

let $\gamma(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ $0 \leq t \leq 2\pi$

$$\int_\gamma f \, ds = \int_0^{2\pi} 1 \, dt = 2\pi$$

$$\neq 0.$$

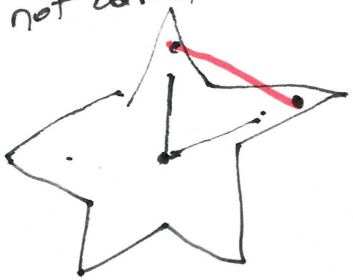
<u>Rk</u>. If f is conservative $\Rightarrow$ sym of partials

$\not\Leftarrow$

Whether the symmetry implies conservative or not depends on where f is defined

If f is defined on a star shaped region then in fact these conditions are also sufficient.

Defn. A sset $X \subset \mathbb{R}^n$ is star shaped if $\exists\, x_0 \in X$ such that $\forall x \in X$, the line segment joining x to x_0 is contained in X.



Convex Any $x, y \in X$ the line segment from x to y is contained in X.

Convex $\Rightarrow$ starshaped

$\not\Leftarrow$

Thm: If X is star shaped open sset of $\mathbb{R}^n$ $f \in C^1$ vector field. Then

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall\, 1 \le i, j \le n$$

imply that f is conservative.

<u>Ex</u>: $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(x,y,z) \longrightarrow \begin{pmatrix} e^x \cos y + yz \\ xz - e^x \sin y \\ xy + z \end{pmatrix}$$

Is f conservative?

Since $\mathbb{R}^3$ is star shaped

we can check if $\operatorname{curl} f = 0$?

$$\underline{\operatorname{curl} f} = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

$$= \begin{pmatrix} x - x \\ y - y \\ \end{pmatrix} \quad (z - e^x \sin y) - (e^x(-\sin y) + z)$$

$$\operatorname{curl} f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

$\Rightarrow f$ is conservative.

$\exists\, g$ such that $f = \nabla g$

$$\frac{\partial g}{\partial x} = e^x \cos y + yz \quad (1)$$

$$\frac{\partial g}{\partial y} = xz - e^x \sin y \quad (2)$$

$$\frac{\partial g}{\partial z} = xy + z. \quad (3)$$

What is g?

$(1) \Rightarrow$

$$g(x,y,z) = e^x \cos y + yzx + h(y,z)$$

$$\frac{\partial g}{\partial y} = -e^x \sin y + zx + \frac{\partial h}{\partial y}(y,z)$$

$$\underset{(2)}{=} xz - e^x \sin y$$

$$\Rightarrow \frac{\partial h}{\partial y}(y,z) = 0$$

$$\Rightarrow h(y,z) = k(z)$$

$$g(x,y,z) = e^x \cos y + xyz + k(z)$$

$$\frac{\partial g}{\partial z} = xy + k'(z) \underset{(3)}{=} xy + z$$

$$\Rightarrow k'(z) = z \Rightarrow k(z) = \frac{z^2}{2} + c$$

$$\Rightarrow \boxed{g(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + c}$$

<u>Next</u>: $f = \mathbb{R}^n \longrightarrow \mathbb{R}$

We will "integrate" f over regions in $\mathbb{R}^n$.

Riemann Integral

for $f:[a,b]\to\mathbb{R}$.

$P = \{x_0 = a < x_1 \cdots < x_n = b\}$ partition of $[a,b]$

$\xi = \{\xi_i\}$, $\xi_i \in [x_{i-1}, x_i]$

Riemann sum $R(f, P, \xi) :=$

$$\sum_{k=1}^{n-1} f(\xi_k)\,\mathrm{Vol}\, I_k$$

where $\mathrm{vol}(I_k) = (x_k - x_{k-1})$

Lower Riemann sum

$$L(P,f) := \sum_{k=1}^{n-1} \left(\operatorname*{Inf}_{I_k} f\right) \mathrm{vol}(I_k)$$

Upper Riemann sum

$$U(P,f) := \sum_{k=1}^{n-1} \left(\sup_{I_k} f\right) \mathrm{Vol}(I_k)$$

Lower Riemann Integral

$$\underline{I}(f) := \operatorname{Sup}\{L(P,f) \mid P \text{ any partition}\}$$

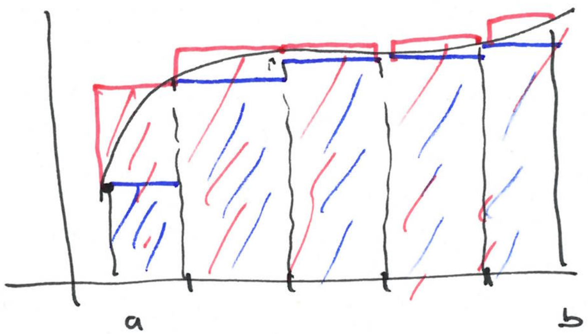
$$\overline{I}(f) := \operatorname{Inf}\{U(P,f) \mid P \text{ any partition}\}$$

Upper Riemann Integral

f is called **Integrable** when

$\overline{I}(f) = \underline{I}(f)$ and we write

$$\int_a^b f\,dx.$$



$U(P,f)$ $L(P,f)$

Thm 1) If f is continuous on $[a,b]$ and bdd then f is integrable.

2) $$\int_a^b f(x) = \lim_{\delta(P_n)\to 0} R(f, P_n, \xi)$$

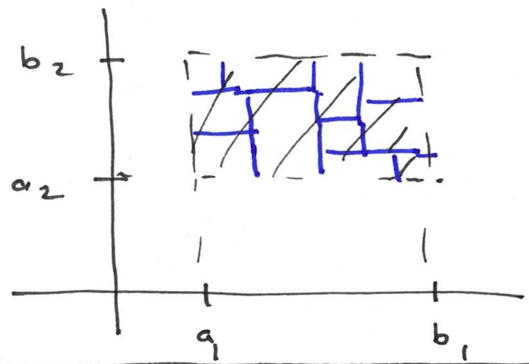
where P_n is a sequence of partitions for which $\delta(P_n) = \max_k (x_k - x_{k-1})$ goes to zero.

$n=2$

$f : Q \to \mathbb{R}$

$Q \subset \mathbb{R}^2$ a rectangle

$Q = I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2]$.



Vol $Q = (b_1 - a_1)(b_2 - a_2)$

A Partition P of Q is a subcollection of rectangular boxes

① $Q = \bigcup_{j=1}^{k} Q_j$

② $\operatorname{Int} Q_i \cap \operatorname{Int} Q_j = \emptyset$ for $i \neq j$

General n : $f : Q \to \mathbb{R}$

$Q \subset \mathbb{R}^n$

$Q = I_1 \times I_2 \times \cdots \times I_n$

$I_k = [a_k, b_k]$

$$\text{Vol } Q = \prod_{i=1}^{n} (b_i - a_i) =: \mu(Q)$$

Norm of $P = \delta_P := \max(\text{vol } Q_j)$

For $\xi_j \in Q_j$, $P = \{Q_j\}_{j=1}^{k}$

$\xi = \{\xi_j\}$

$$R(f, P, \xi) := \sum_{j=1}^{k} f(\xi_j)(\text{Vol } Q_j)$$

Riemann Sum of f for Partition P.

Lower Riemann Sum

$$L(P,f) := \sum_{j=1}^{k} \left(\inf_{Q_j} f\right) \mathrm{vol}(Q_j)$$

Upper Riemann Sum

$$U(P,f) := \sum_{j=1}^{k} \left(\sup_{Q_j} f\right) \mathrm{vol}(Q_j)$$

Lower Riemann Integral

$$\underline{I}(f) := \sup \{ L(P,f) \mid P \text{ partition of } Q \}.$$

Upper Riemann Integral

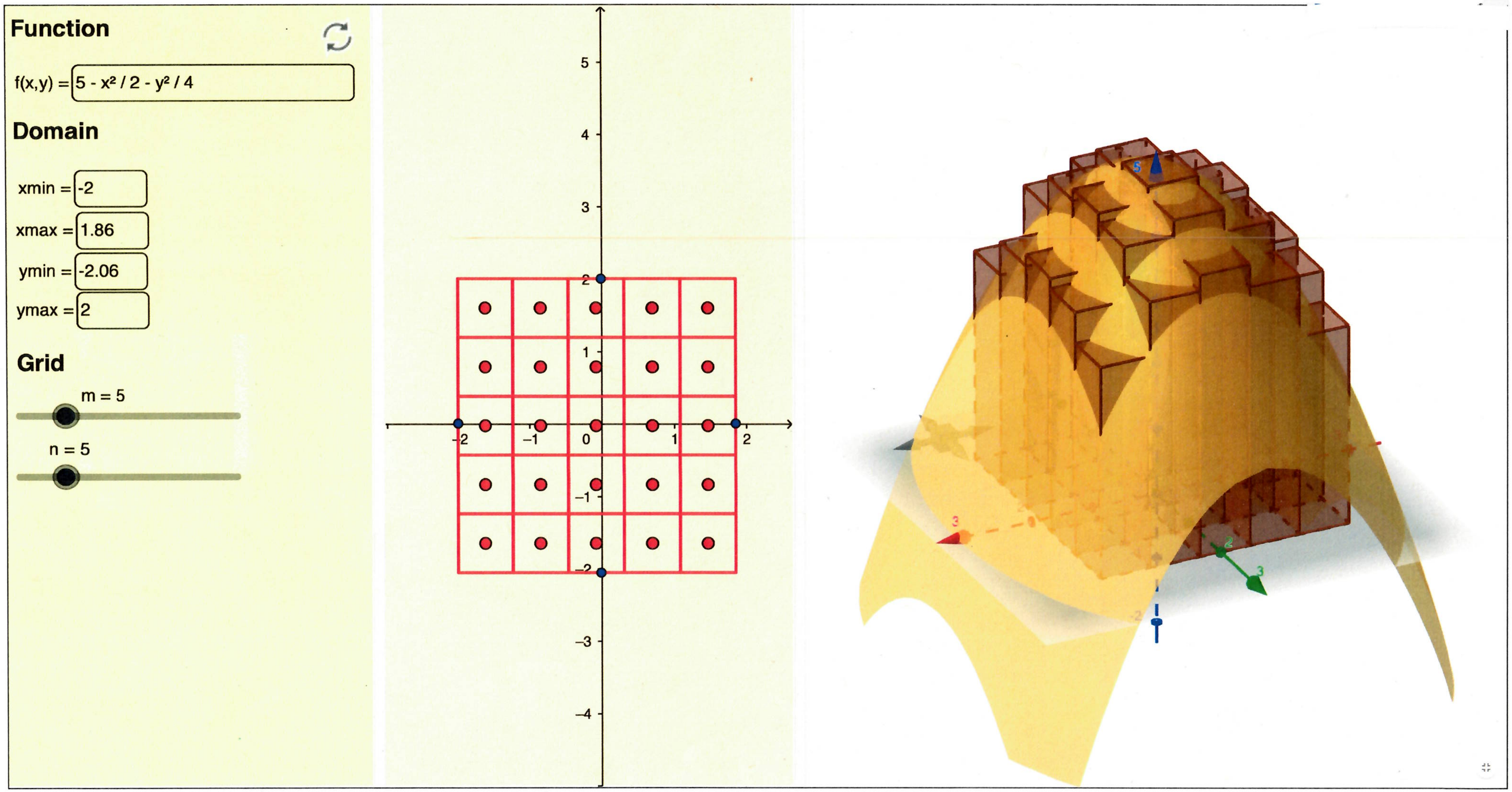
$$\overline{I}(f) := \inf \{ U(P,f) \mid P \}.$$

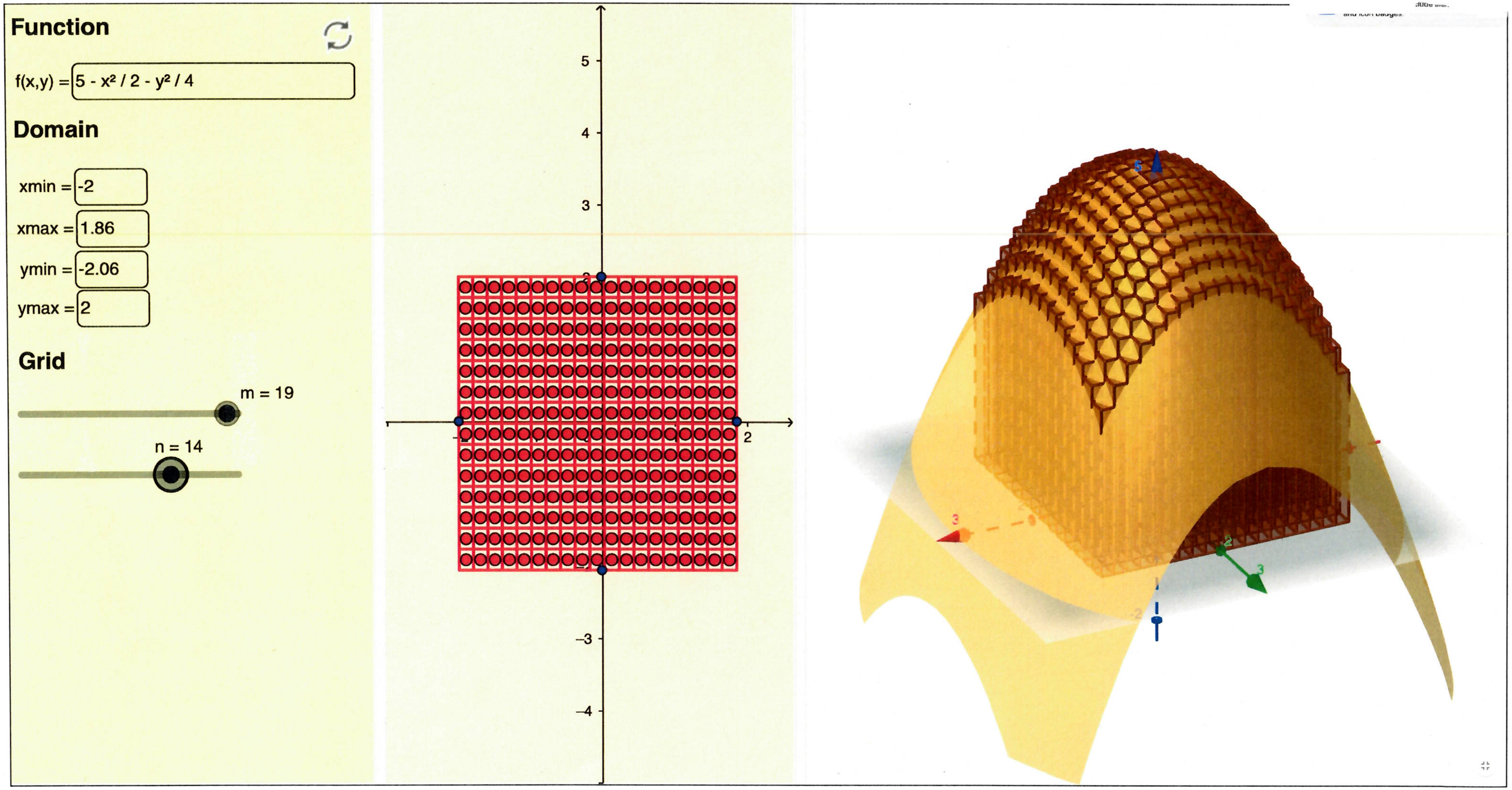
Defn $f : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^n$ is called (R-) integrable if

$$\boxed{\underline{I}(f) = \overline{I}(f).}$$

and we write

$$\int_Q f \, d\underline{x} = \int_Q f(x_1, \ldots x_n) \, dx_1 \ldots dx_n.$$


Function
f(x,y) = 5 - x² / 2 - y² / 4
Domain
xmin = -2
xmax = 1.86
ymin = -2.06
ymax = 2
Grid
m = 5
n = 5


Function
f(x,y) = 5 - x² / 2 - y² / 4
Domain
xmin = -2
xmax = 1.86
ymin = -2.06
ymax = 2
Grid
m = 19
n = 14
5
4
3
2
−3
−4

$Q \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

Thm $f, g : [a,b] \longrightarrow \mathbb{R}$ integrable, $\alpha, \beta \in \mathbb{R}$. Then

1) $\alpha f + \beta g$ is also integrable and
$$\int (\alpha f + \beta g)\, dx = \alpha \int f\, dx + \beta \int g\, dx$$

2) if $f(x) \le g(x)$ $\forall x \in [a,b]$ then
$$\int_{\substack{[a,b] \\ Q}} f\, dx \le \int_{\substack{[a,b] \\ Q}} g\, dx$$

3) if $f(x) \ge 0$ then $\displaystyle\int_{\substack{[a,b] \\ Q}} f\, dx \ge 0$.

4) $$\left| \int_{[a,b]} f\, dx \right| \le \int_{[a,b]} |f|\, dx \le (\sup f)(\mathrm{vol}\,[a,b])$$

$$\left| \int_Q f\, dx \right| \le \int_Q |f|\, dx \le \left(\sup_Q f\right)(\mathrm{vol}\, Q)$$

5) Fubini's theorem.

if $Q = I_1 \times I_2 \times \ldots \times I_n$
$[a_1, b_1] \times \ldots [a_n, b_n]$

$$\int_Q f\, d\underline{x} = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \cdot \left(\cdot \left(\int_{a_n}^{b_n} f(x_1 \ldots x_n)\, dx_n \right) dx_{n-1} \right) \ldots \right) dx_1$$

Eg. $Q = [1,2]\times[2,4]$.

$f(x,y) = x^2+y^2$

$$\int_Q (x^2+y^2)\,dx\,dy$$

$$= \int_1^2 \left(\int_2^4 (x^2+y^2)\,dy\right) dx$$

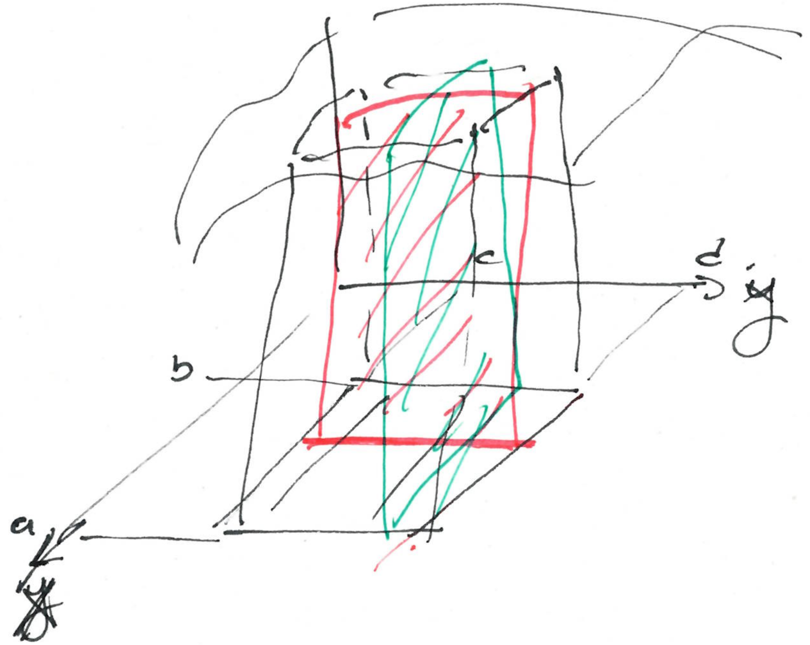
$$= \int_1^2 \left(x^2y+\frac{y^3}{3} \Big|_2^4 \right) dx$$

$$= \int_1^2 \left(4x^2+\frac{4^3}{3}\right)-\left(2x^2+\frac{8}{3}\right) dx$$

$$= \int_1^2 2x^2+\left(\frac{4^3}{3}-\frac{8}{3}\right) dx = \cdots$$

Thm f is continous and bounded on Q then f is integrable.

$f: \mathbb{R}^2 \to \mathbb{R}$.



Order of integration:

☐ $dA = dy\,dx$

☑ $dA = dx\,dy$

Region of integration:

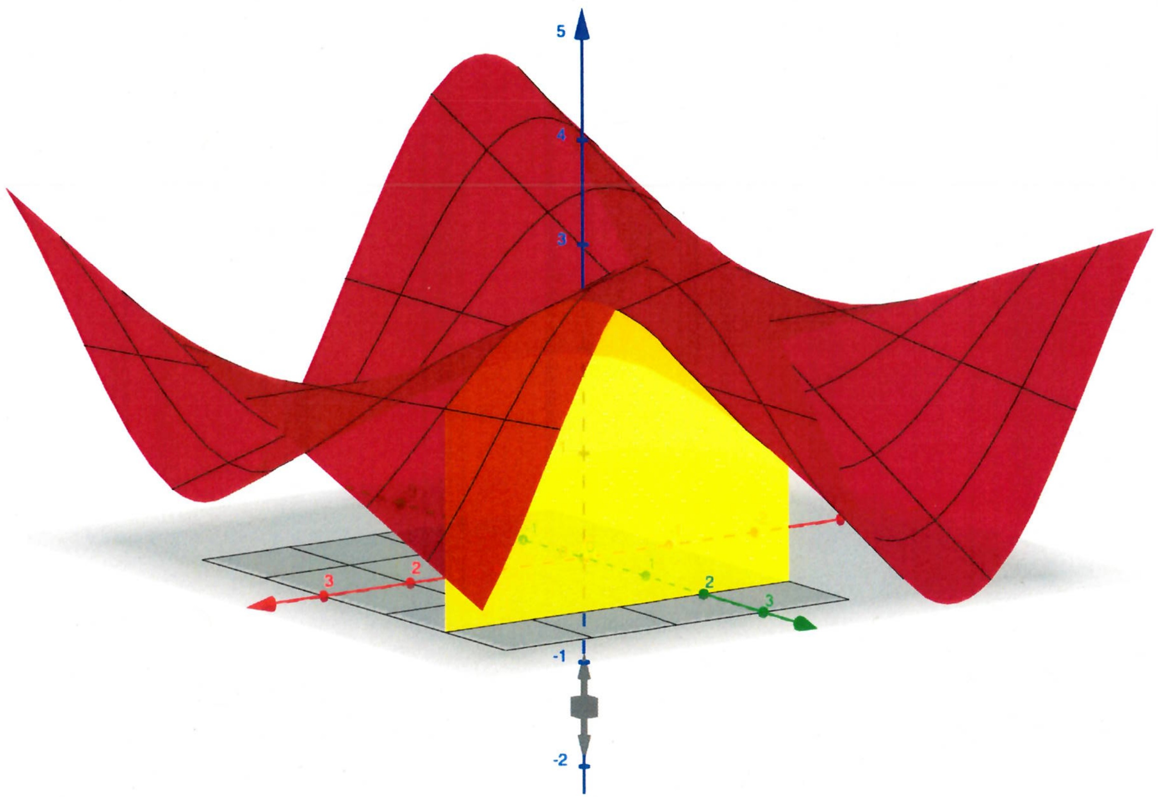
$$\left.\begin{array}{l} \boxed{-1} \le x \le \boxed{3} \\ \boxed{-2} \le y \le \boxed{3} \end{array}\right\}\mathcal{D}$$

$y_0 = 2$

$$\iint\limits_{\mathcal{D}} f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_c^d A(y)\,dy$$

$A(y_0)$ = area of yellow region



Order of integration:

- [x] $dA = dy\,dx$
- [ ] $dA = dx\,dy$

Region of integration:

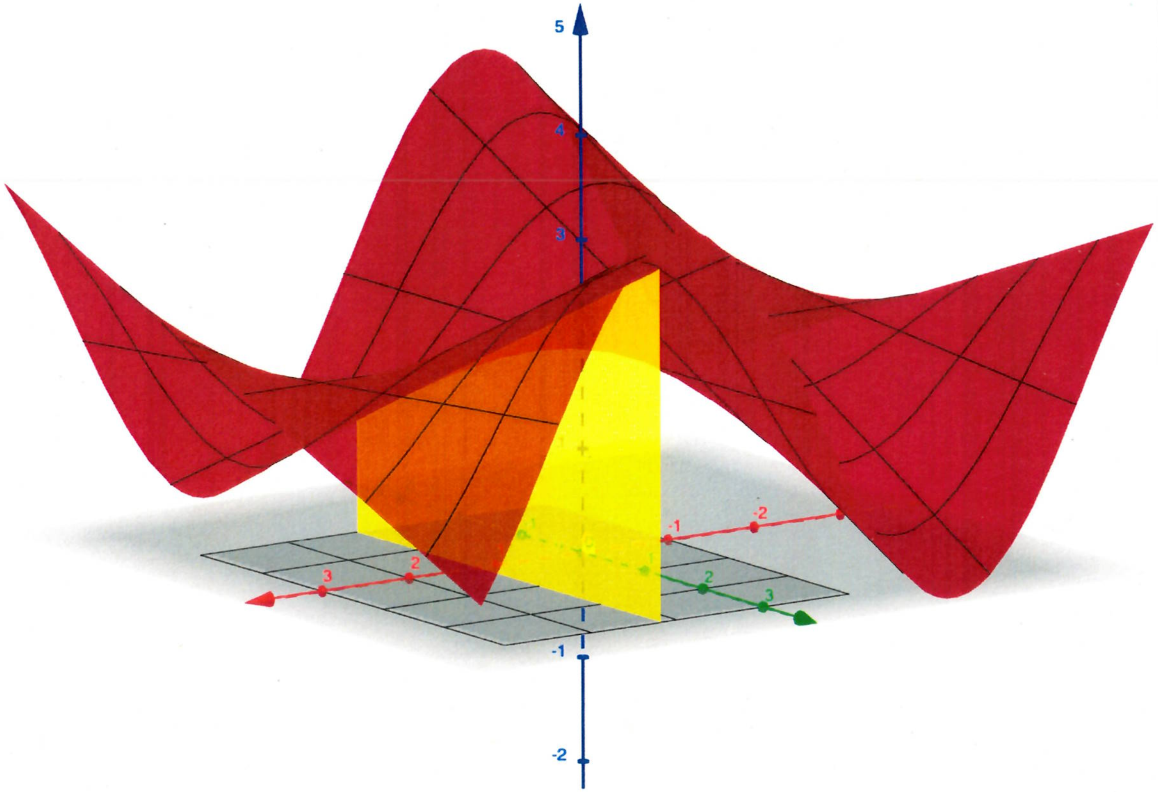
$$\left.\begin{array}{l} \boxed{-1} \leq x \leq \boxed{3} \\ \boxed{-2} \leq y \leq \boxed{3} \end{array}\right\} \mathcal{D}$$

$x_0 = 1.2$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

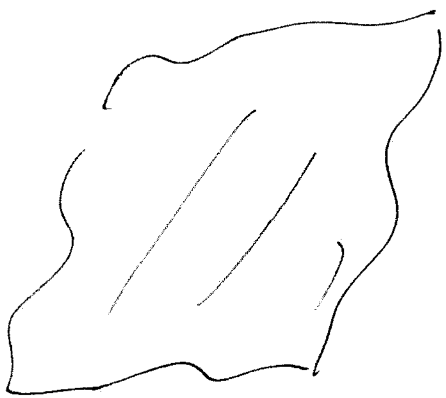
$$= \int_a^b A(x)\,dx$$

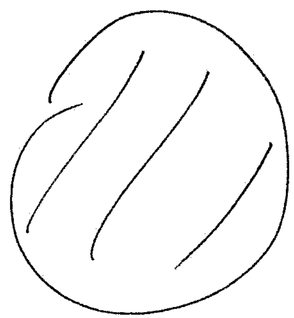
$A(x_0) =$ area of yellow region

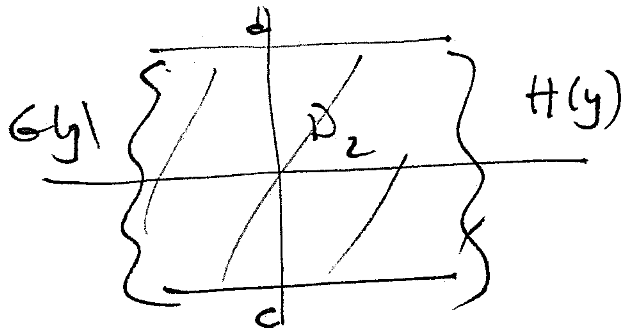


In $\mathbb{R}^n$ we have more complicated regions than

$$I_1 \times I_2 \times \cdots I_n$$

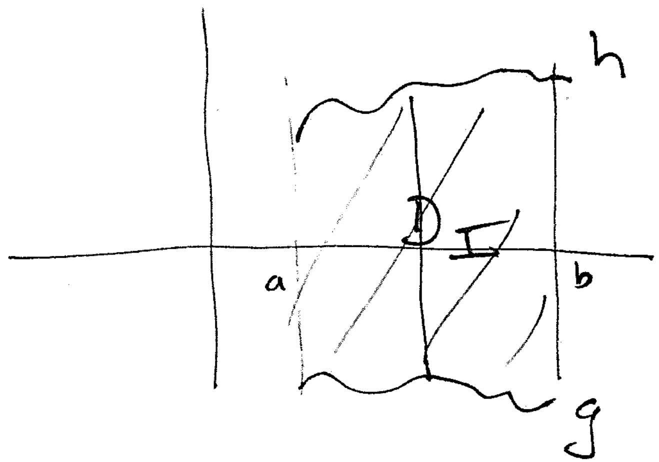




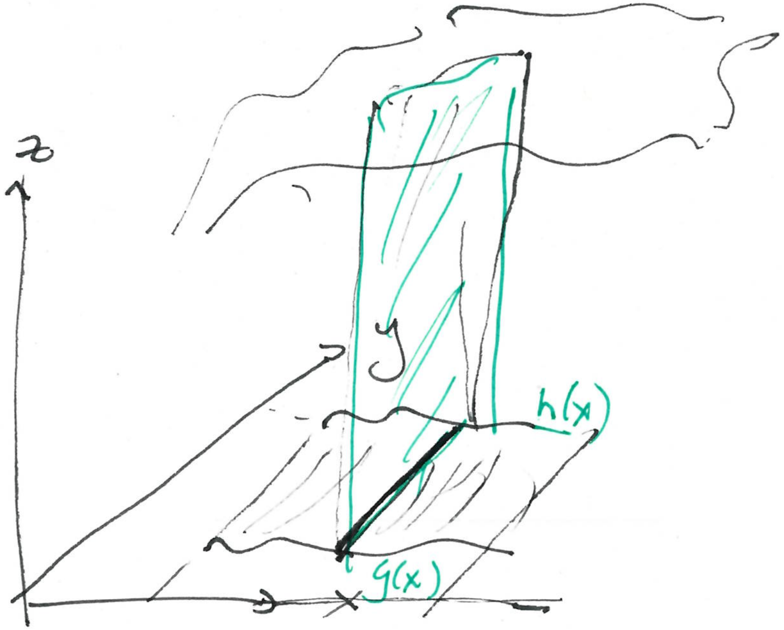


Suppose $D \subset \mathbb{R}^2$

$$D_I := \{ (x,y) \mid a \le x \le b,\ g(x) \le y \le h(x) \}$$



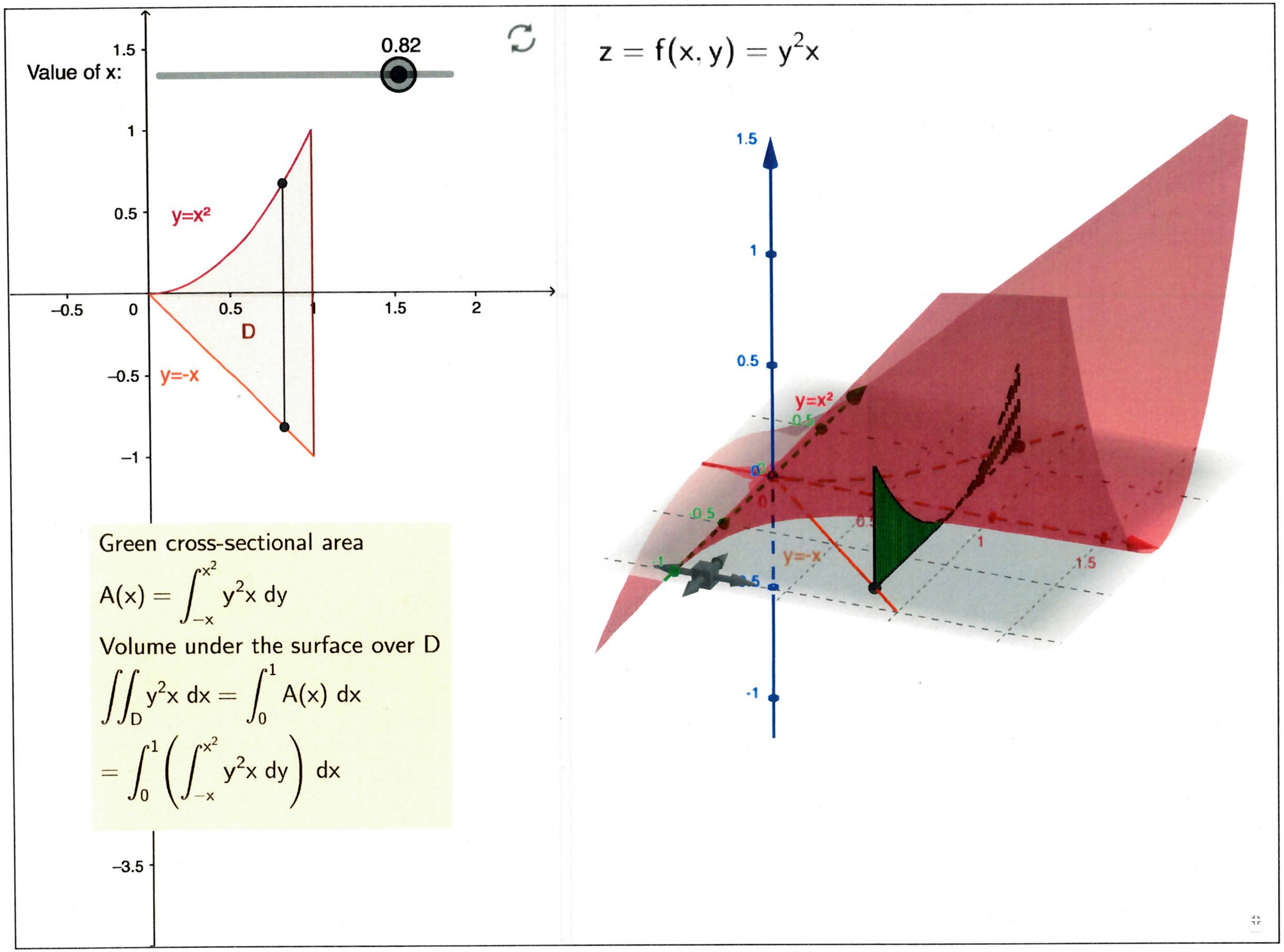
$$D_{II} := \{ (x,y) \mid c \le y \le d,\ G(y) \le x \le H(y) \}$$

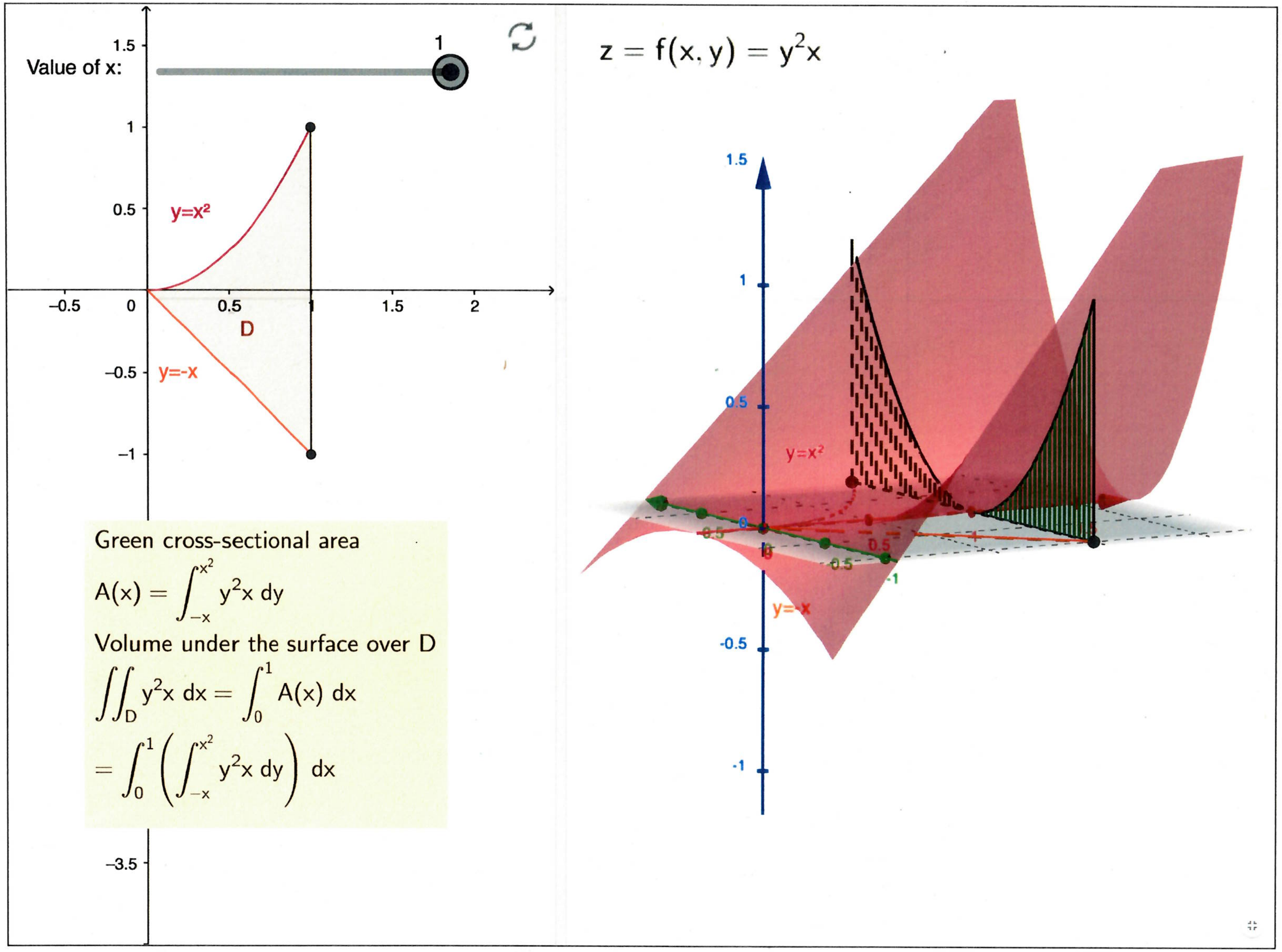


Cross section has area.

$$\Delta A = \int_{g(x)}^{h(x)} f(x,y)\,dy$$

$$\sum \Delta A = \int_a^b \left(\int_{g(x)=y}^{h(x)=y} f(x,y)\,dy \right) dx.$$


Value of x:
0.82
$z = f(x, y) = y^2x$
y=x²
y=-x
D
Green cross-sectional area
$A(x) = \int_{-x}^{x^2} y^2x \, dy$
Volume under the surface over D
$\iint_D y^2x \, dx = \int_0^1 A(x) \, dx$
$= \int_0^1 \left(\int_{-x}^{x^2} y^2x \, dy \right) dx$

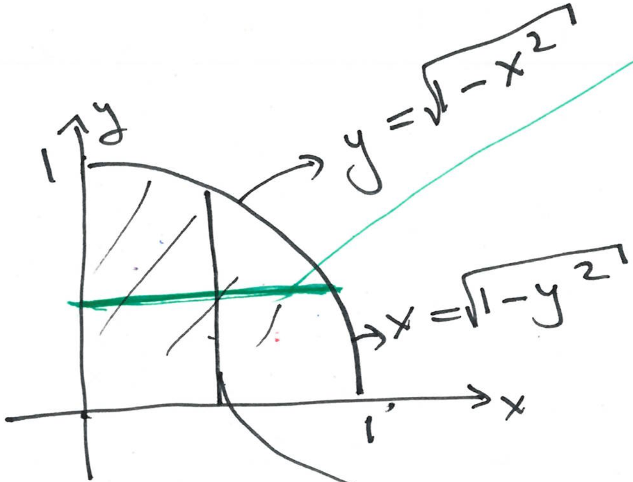

Value of x:
1
y=x²
y=-x
D
Green cross-sectional area
$A(x) = \int_{-x}^{x^2} y^2 x \, dy$
Volume under the surface over D
$\iint_D y^2 x \, dx = \int_0^1 A(x) \, dx$
$= \int_0^1 \left(\int_{-x}^{x^2} y^2 x \, dy \right) dx$
$z = f(x, y) = y^2 x$

Examples

$x^2+y^2=1$

$f(x,y) = x+y$

D = quarter of the unit circle



$$\int_D f\,dx\,dy = ?$$

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} \left((x+y)\,dx \right) \right) dy$$

$$\int_0^1 \left(\int_{0}^{\sqrt{1-x^2}} (x+y)\,dy \right) dx$$

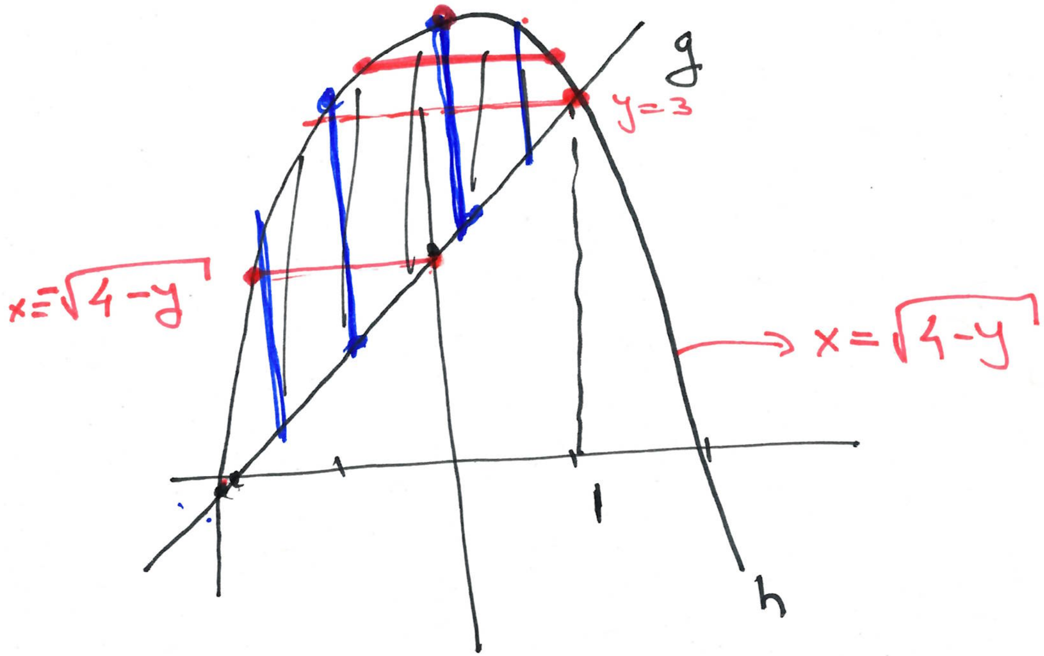
$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} (x+y)\,dx \right) dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + yx \Big|_{x=0}^{x=\sqrt{1-y^2}} \right) dy$$

$$= \int_0^1 \left(\frac{(1-y^2)}{2} + y \cdot \sqrt{1-y^2} \right) dy .$$

2) $f(x,y) = x$

D is the region bounded by the line $g(x) = x+2$ and the parabola $h(x) = 4 - x^2 = y$



Intersection points:

$$x+2 = 4 - x^2$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2)$$

$$x = -2$$

$$x = 1$$

$$\int_{-2}^{1} \left(\int_{y=x+2}^{y=4-x^2} x \, dy \right) dx$$

$$= \int_{-2}^{1} \left(xy \Big|_{y=x+2}^{4-x^2=y} \right) dx$$

$$= \int_{-2}^{1} \left[x(4-x^2) - x(x+2) \right] dx = \ldots$$

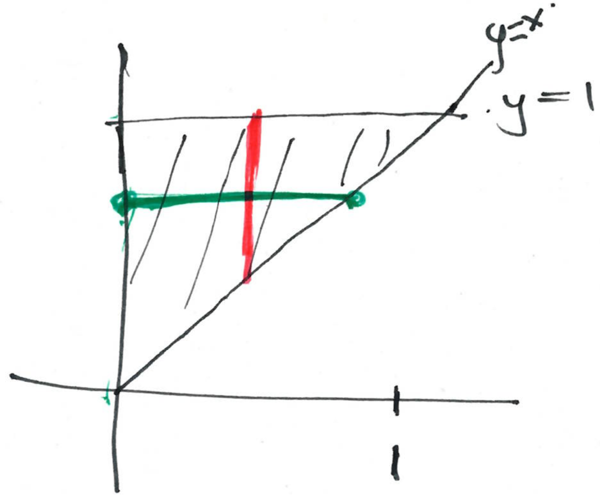
$$\int f \, dx \, dy$$

$$= \int_0^3 \int_{-\sqrt{4-y}}^{y-2} x \, dx \, dy$$

$$+ \int_3^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} x \, dx \, dy$$

3) $$\boxed{\int_0^1 \left(\int_x^1 e^{y^2} dy \right) dx = ?}$$

What is the region of integration?



$\int e^{y^2} dy$ does not have an explicit primitive

Instead we'll try and see if we can change order of integration and then integrate.

$$\int_0^1 \left(\int_0^y e^{y^2}\,dx \right) dy$$

$$\int_0^1 \left(e^{y^2} \cdot x \Big|_0^y \right) dy$$

$$= \int_0^1 y e^{y^2}\,dy = \frac{e^{y^2}}{2} \Big|_0^1 = \frac{1}{2}(e-1)$$

Moral of the story:

Sometimes one has to change the order of integration to be able to calculate the integral!