Riemann Integral

$Q = I_1 \times I_2 \times \dots \times I_n \subset \mathbb{R}^n$

$f: Q \longrightarrow \mathbb{R}$, $\operatorname{vol} Q_k = \prod \operatorname{vol}(I_k)$

$I_k = [a_k, b_k]$, $\operatorname{Vol}(I_k) = (b_k - a_k)$

P = a partition of $Q = \{Q_1, Q_2, \dots Q_k\}$
= a collection of rectangular boxes
s.t. ① $Q = \bigcup_{j=1}^{k} Q_j$

② $\operatorname{Interior}(Q_j) \cap \operatorname{Interior} Q_i = \phi$ for $i \neq j$

$\operatorname{Norm}(P) =: \delta_P := \max(\operatorname{vol} Q_j)$

For $\xi_j \in Q_j$, we define the Riemann sum of f for partition P,

$$\boxed{R(f, P, \xi) := \sum_{j=1}^{k} f(\xi_j) \operatorname{vol}(Q_j)}$$

$\xi = \{\xi_j\}_{j=1}^{k}$

Lower Riemann sum

$$L(P, f) := \sum_{j=1}^{k} \left(\inf_{Q_j} f\right)(\operatorname{Vol} Q_j)$$

Upper Riemann sum

$$U(P, f) := \sum_{j=1}^{k} \left(\sup_{Q_j} f\right)(\operatorname{vol} Q_j)$$

Lower Riemann Integral

$$\underline{I}(f) := \sup\{L(P, f) \mid P \text{ partition of } Q\}$$

Upper Riemann Integral

$$\overline{I}(f) := \operatorname{Inf}\{U(P, f) \mid P \text{ partition of } Q\}$$

Defn: $f: Q \longrightarrow \mathbb{R}$ is called (Riemann)-Integrable if

$\underline{I}(f) = \overline{I}(f)$ and

we write $\int_Q f\,dx = \int_Q f(x_1, \dots x_n)\,dx_1 \dots dx_n$

Thm 1) If f is continuous and bounded on Q then f is integrable.

2) $f, g : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^n$ integrable, $\alpha, \beta \in \mathbb{R}$ then

$\alpha f + \beta g$ is integrable

$$\int_Q (\alpha f + \beta g)\, dx = \alpha \int_Q f\, dx + \beta \int_Q g\, dx$$

3) If $f(x) \le g(x)$ $\forall x \in Q$ then

$$\int_Q f\, dx \le \int_Q g\, dx$$

4) If $f(x) \ge 0$ then $\int_Q f\, dx \ge 0$

5)
$$\left| \int_Q f\, dx \right| \le \int_Q |f|\, dx \le (\sup_Q f)\,(\mathrm{vol}\, Q)$$

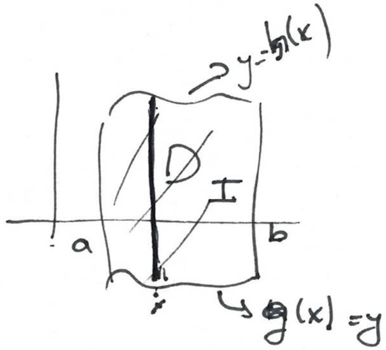
6) **Fubini's theorem**

If $Q = I_1 \times \dots \times I_n = [a_1, b_1] \times \dots [a_n, b_n]$

then
$$\int_Q f\, dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdot \left(\left(\int_{a_n}^{b_n} f(x)\, dx_n \right) dx_{n-1} \right) \dots dx_1$$

Fubini's theorem for general regions

$n=2$ $\quad D_I := \{(x,y) \mid a \le x \le b, \; g(x) \le y \le h(x)$



$$\boxed{\int_{D_I} f\, dx\, dy = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx}$$

$D_{II} := \{(x,y) \mid c \le y \le d, \; G(y) \le x \le H(y)$

$$\boxed{\int_{D_{II}} f\, dx\, dy = \int_c^d \int_{G(y)}^{H(y)} f(x,y)\, dx\, dy .}$$

General Fubini

$X \subset \mathbb{R}^n$, $\quad f : X \to \mathbb{R}$

$n = n_1 + n_2$, $\quad n_1, n_2 \geq 1$

For $x \in \mathbb{R}^n$ write $x = (x_1, x_2)$

with $x_1 \in \mathbb{R}^{n_1}$, $\quad x_2 \in \mathbb{R}^{n_2}$

For $x_1 \in \mathbb{R}^{n_1}$, define

$$X_{x_1} := \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$$

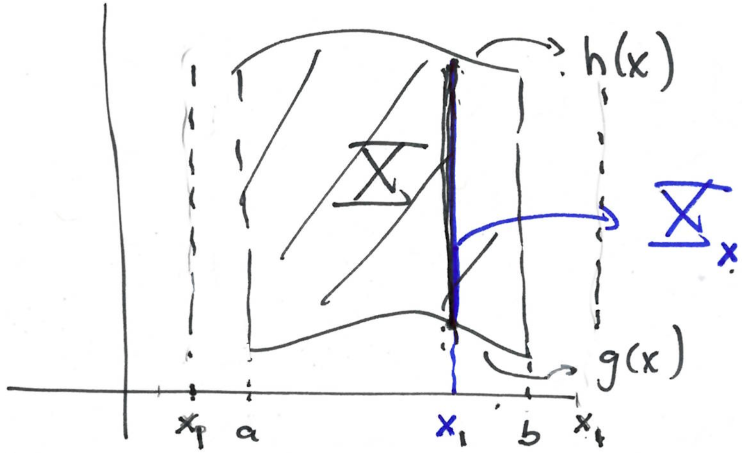
Define $X_1 := \{x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset\} \subset \mathbb{R}^{n_1}$

If $g(x_1) := \int_{X_{x_1}} f(x_1, x_2)\, dx_2$ is continous on X_1

then

$$\boxed{\int_X f(x)\, dx = \int_{X_1} g(x_1)\, dx_1 = \int_{X_1} \int_{X_{x_1}} f(x_1, x_2)\, dx_2\, dx_1}$$

Eg $\quad X = \{(x,y) \mid a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$



If $x_1 \in [a, b]$, then

$$\emptyset \neq X_{x_1} := \{y \in \mathbb{R} \mid (x, y) \in X\}$$
$$= \{y \in \mathbb{R} \mid g(x) \leq y \leq h(x)\}$$

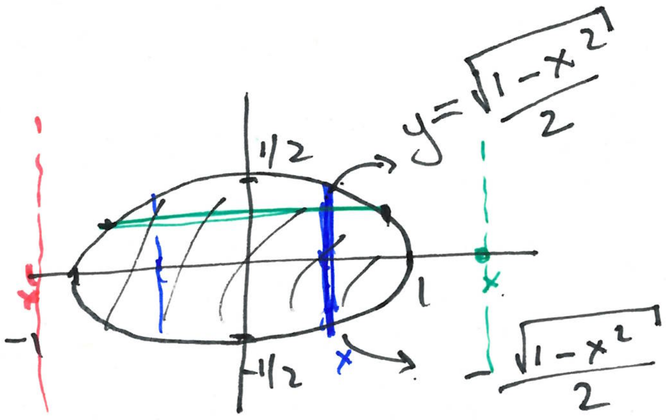
If $x > b$ or $x < a$

then $X_{x_1} = \emptyset$

Hence $X_1 = [a, b]$

$$\int_X f\, dx\, dy = \int_a^b \int_{X_x} f(x,y)\, dy\, dx$$

Eg $\quad X = \{(x,y) \mid x^2 + 4y^2 \leq 1\}$.



For fixed x

$$X_x := \{y \in \mathbb{R} \mid (x,y) \in X\}.$$

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$$= \phi$$

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$$= \phi$$

if $-1 \leq x \leq 1$

$$X_x = \{y \in \mathbb{R} \mid (x,y) \in X\} \neq \phi$$

$$X_1 := \{x \in \mathbb{R} \mid X_x \neq \phi\}$$

$$= [-1, 1].$$

For a given x, $-1 \leq x \leq 1$

$$X_x = \{y \in \mathbb{R} \mid (x,y) \in X\}.$$

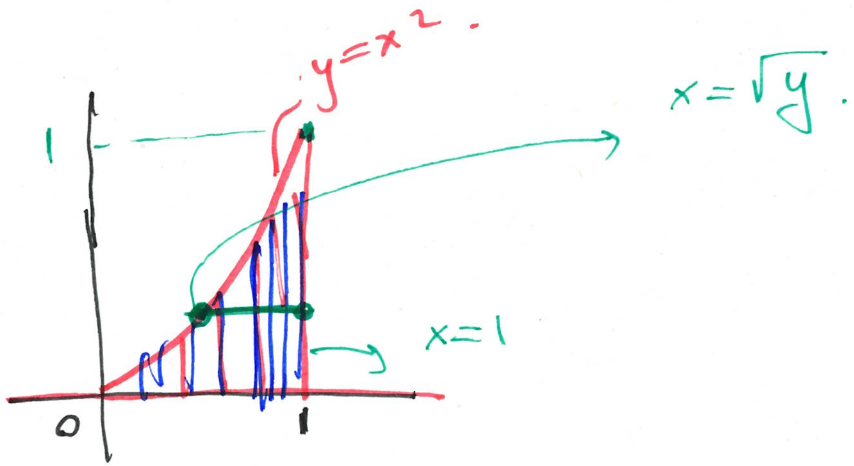
$$= \left\{y \in \mathbb{R} \;\middle|\; -\frac{\sqrt{1-x^2}}{2} \leq y \leq \frac{\sqrt{1-x^2}}{2}\right\}.$$

$$\int_X f = \int_{-1}^{1} \int_{-\frac{\sqrt{1-x^2}}{2}}^{\frac{\sqrt{1-x^2}}{2}} dy\, dx$$

$$= \int_{X_1} \left(\int_{X_x} f(x,y)\, dy \right) dx$$

We could also write it as

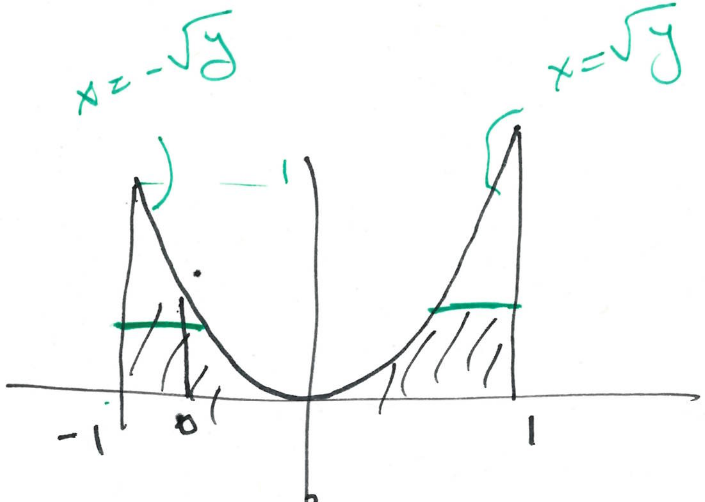
$$\int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} f(x,y)\,dx\,dy$$



Clicker

$$\boxed{\int_0^1\int_0^{x^2} f(x,y)\,dy\,dx}$$

$$\int_0^1\int_{\sqrt{y}}^1 f(x,y)\,dx\,dy$$

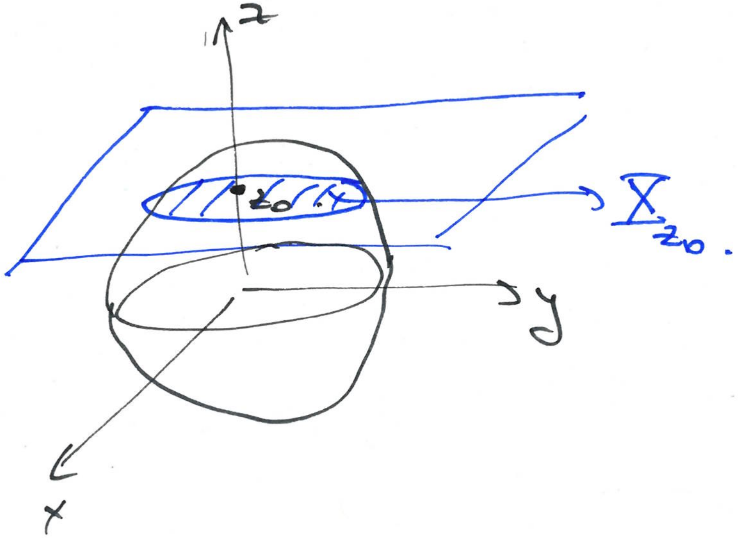


$$\int_{-1}^1\int_0^{x^2} f(x,y)\,dy\,dx$$

$$=\int_0^1\int_{-1}^{-\sqrt{y}} f\,dx\,dy+\int_0^1\int_{\sqrt{y}}^1 f\,dx\,dy$$

$n=3$, $2+1$ let $f: \mathbb{R}^3 \to \mathbb{R}$
$(x,y,z) \to 1$

Ex $\quad X = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$.



Fix $z = z_0$

$$X_{z_0} := \{(x,y,z_0) \mid (x,y,z_0) \in X\}.$$

$$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1-z_0^2\}$$

If $z_0 > 1$ or $z_0 < -1$

$$X_{z_0} = \emptyset$$

$$\int_{-1}^{1} \underbrace{\left(\iint_{X_z} 1 \, dx\,dy\right)}_{\substack{\text{area of } X_z \\ = \text{disc of radius } \sqrt{1-z^2}. \\ = \pi(1-z^2)}} dz$$

$$= \int_{-1}^{1} \pi(1-z^2)\,dz = \pi\left(z - \frac{z^3}{3}\Big|_{-1}^{1}\right)$$

$$2\pi\left(1-\frac{1}{3}\right) = \frac{4\pi}{3}$$

Remark

In Fubini's thm.

we have the assumption

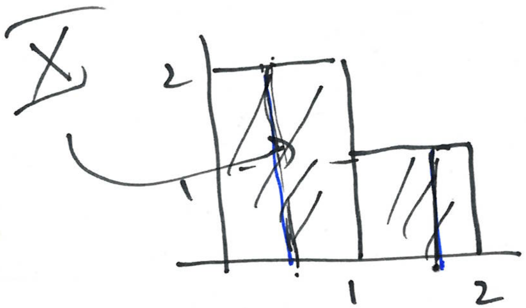
$$g(x_1) := \int_{X_{x_1}} f(x_1, x_2)\, dx_2$$

is continuous on X_1

This is not always the case!!

Eg. $n=2$ $f=1$

$\int_X 1\, dx\, dy$ exists



$X_1 = \emptyset$ if $x>2$ or $x<0$.

$\neq \emptyset$ if $0 \leq x \leq 2$.

$X_1 = [0,2]$.

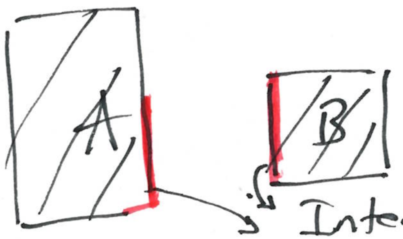
$$X_x = \begin{cases} [0,2] & x \leq 1 \\ [0,1] & 2 \geq x \geq 1 \end{cases}$$

$$g(x) = \int_{X_x} 1\, dy = \begin{cases} 2 & \text{if } x \leq 1 \\ 1 & \text{if } 1 \leq 2 \leq 2 \end{cases}$$

Is not continuous.

We can work around this issue by dividing the region $\bar{X}$ into pieces

$A \cup B = \bar{X}$



Integrating twice but they're negligable

$$\int_A 1\, dx\, dy \qquad \int_B 1\, dx\, dy.$$

1-dim'l integral

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$$

a pt $\{c\}$ is negligible.

<u>Defn</u> 1) For $1 \leq m \leq n$ a <u>m-parametrized set</u> or parametrized m-set is a continous function

$$\varphi : [a_1, b_1] \times \cdots [a_m, b_m] \to \mathbb{R}^n$$

which is C^1 on $(a_1, b_1) \times \cdots (a_m, b_m)$.

If $m=1$, then φ is a parametrized curve in $\mathbb{R}^n$

2) A sset $Y \subset \mathbb{R}^n$ is called <u>negligible</u> if $\exists$ finitely many φ_i, parametrized m_i-sets with $m_i < n$

such that $1 \le i \le k$

$$Y \subset \bigcup_{i=1}^{k} \varphi_i(X_i)$$

where $\varphi_i : X_i \to \mathbb{R}^n$.

<u>$n=1$</u> union of finitely many points are negligible.

<u>$n=2$</u> Union of finitely many images of parametrized curves and finitely many points

<u>$n=3$</u> union of finitely many images of parametrized surfaces, curves, points

<u>Thm</u> If $Y \subset \mathbb{R}^n$ is negligible closed bounded then

$$\int_Y f(x_1, \dots x_n)\, dx_1 \dots dx_n = 0.$$

$\forall\ f : Y \to \mathbb{R}$ continuous.

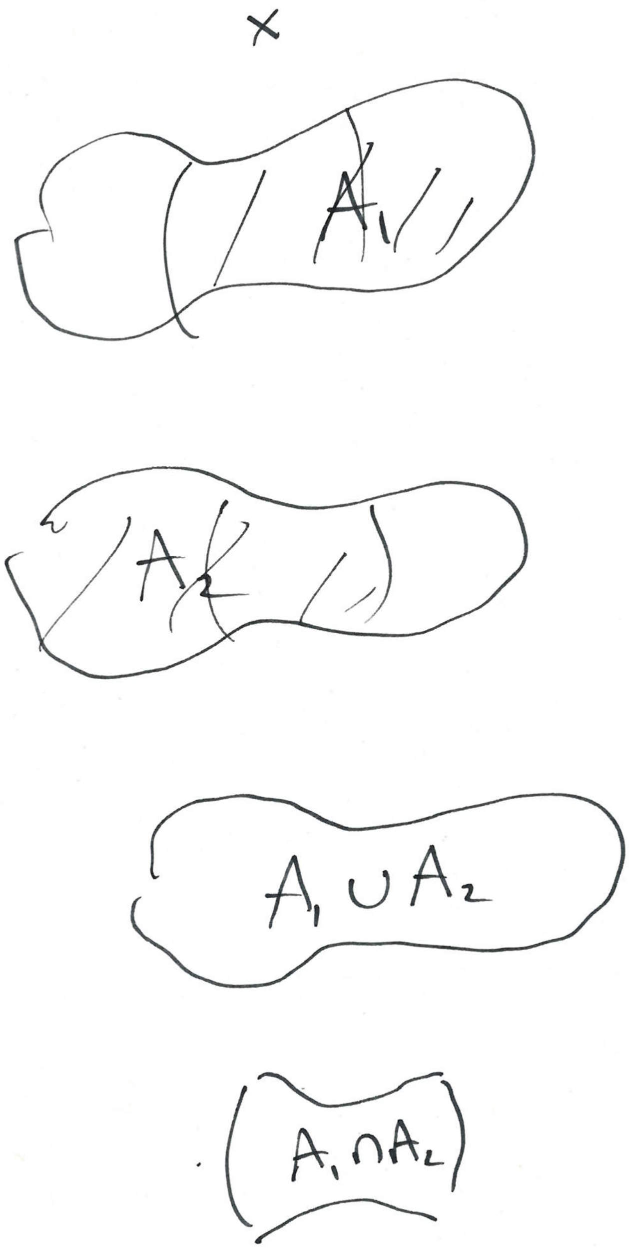
Important property of the integral is the domain additivity.

If $X = A_1 \cup A_2$

A_1, A_2 bounded and closed then

for $f : X \to \mathbb{R}$

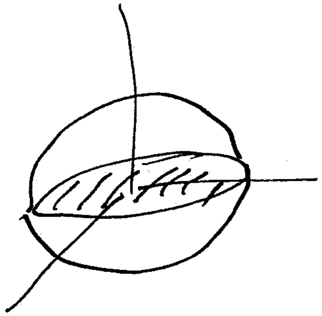
$$\boxed{\int_X f\,dx = \int_{A_1} f\,dx + \int_{A_2} f\,dx - \int_{A_1 \cap A_2} f\,dx}$$



In the special case that $A_1 \cap A_2$ is negligible in $\mathbb{R}^n$

we have

$$\int_X f\,dx = \int_{A_1} f\,dx + \int_{A_2} f\,dx_2$$



$A_1 \cap A_2 \subset xy$ plane is the disc $x^2+y^2 \le 1$ is negligible.

Ex $X = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$.

$A_1 = \{(x,y,z) \cancel{\in} \; z \ge 0\}$

$A_2 = \{(x,y,z) \in X \mid z \le 0\}$

$$\int_X 1\,dx\,dy\,dz = \int_{A_1} + \int_{A_2} - \int_{A_1 \cap A_2}$$

$0 =$

$$A_1 = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x^2+y^2 \le 1, \; 0 \le z \le \sqrt{1-x^2-y^2}\}.$$

$$\int_{A_1} 1 \, dx\,dy\,dz = \int_D \sqrt{1-x^2-y^2}\, dx\,dy \longleftarrow$$

where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$.

If we use 2 dim'l Fubini to evaluate the integral $\int_D \sqrt{1-x^2-y^2}\, dx\,dy$

$$= \int_{-1}^{1} g(x)\, dx$$

where

$$g(x) = \underbrace{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy}$$

This integral is of the form

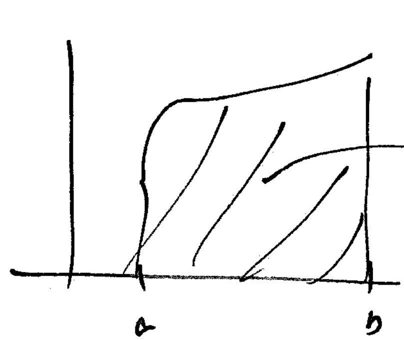
$$\int_{-R}^{R} \sqrt{R^2-y^2}\, dy = \text{area of half of a disc of radius } R$$

$$\int_D \sqrt{1-x^2-y^2}\, dx\,dy = \frac{\pi}{2} \int_{-1}^{1} (1-x^2)\, dx = \frac{2\pi}{3}.$$

Rk Here we used the following geometric interpretation of the integral

$n=1$, $f \ge 0$ $\quad \int_c^b f \, dx$

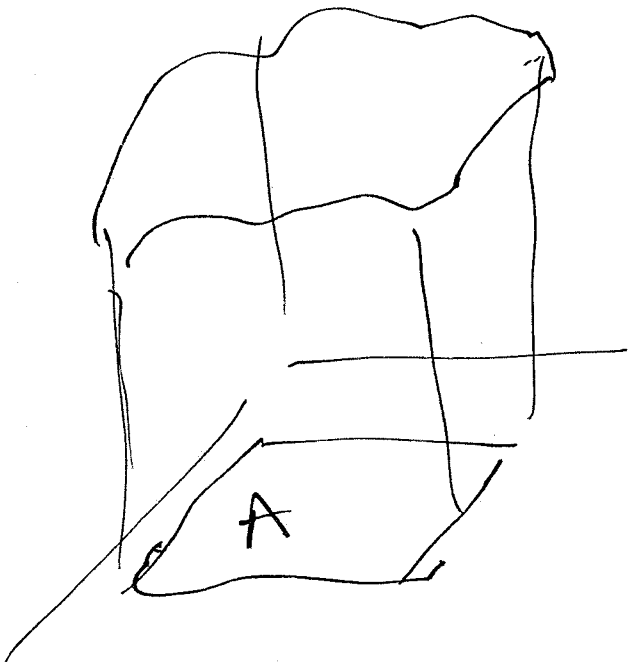
gives the area under the graph of f between a and b



$$\longrightarrow \int_c^b f \, dx$$

For general n, if $A \subset \mathbb{R}^n$

$f : A \to \mathbb{R}$.



$$\int_A f\,dx = \text{volume under the graph of } f \text{ over the region } A.$$

In the particular case

$f \equiv 1$

$$\int_A 1\,dx = \text{area of } A.$$

Recall $n=1$

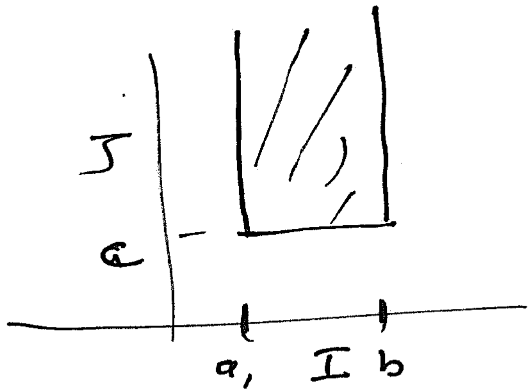
we could extend the defn of integral to unbounded domains or to functions that are unbounded themselves.

$$\int_a^\infty \frac{1}{x^2}\,dx \qquad \int_0^1 \frac{1}{x^2}\,dx$$

let $I \subset \mathbb{R}$ bounded interval

$J = [a, \infty)$

$f : I \times J \to \mathbb{R}$ continuous.



We say f is integrable on $I \times J$ if

$$\lim_{B\to\infty} \int_a^B \int_I f(x,y)\,dx\,dy = \lim_{B\to\infty} \int_I \int_a^b f\,dy\,dx$$

exists.

and we denote the limit

with $\displaystyle\int_a^\infty \int_I f \, dx \, dy$

$$= \int_{I \times J} f \, dx \, dy.$$

<u>Ex</u> $X = [a, b] \times [c, \infty)$

(a) $f = 1$

$$\int_{[a,b]} \int_{[c,B]} 1 \, dx \, dy = (b-a)(B-c)$$

as $B \to \infty$ does not converge.

$\int f$ doesn't converge, f is not integ.

b) $f = \frac{1}{y^2}$ $\quad \displaystyle\int_{I \times J} \frac{1}{y^2} \, dx \, dy$ conv.

$$\lim_{B \to \infty} \int_c^B \int_a^b \frac{1}{y^2} \, dx \, dy.$$

$$= \lim_{B \to \infty} \int_c^B \left(\frac{x}{y^2} \Big|_a^b \right) dy.$$

$$(b-a) \underbrace{\lim_{B \to \infty} \int_c^B \frac{1}{y^2} \, dy}_{\lim_{B \to \infty} \frac{-1}{y} \Big|_c^B}.$$

$$= \frac{b-a}{c} \quad \text{converges.}$$

In general

let $X \subset \mathbb{R}^n$ be a non-compact set

$f : X \to \mathbb{R}$ such that

$\int_K f \, dx$ exists

(closed, bdd)

for every K compact

sset. of X

Suppose we have a sequence of regions X_n

such that 1) $X_n \subset X$ bdd closed

2) $X_n \subset X_{n+1}$

③ $\bigcup X_n = X$

eg. $n = 1$

$X_n = [-n, n]$

$\bigcup X_n = \mathbb{R}$.

If $\lim_{n \to \infty} \int_{X_n} f$ exists

then we say $\int_X f \, dx$

converges.

eg. $n=2$

we can choose

X_n as expanding squares

or discs . . .

Ex: $X = \mathbb{R}^2$

$$\int_X e^{-x^2-y^2}\,dx\,dy = ?$$

$$= \int_{\mathbb{R}^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$$

To be able to evaluate this integral, we want to talk about change of variables.

§4.4 Change of variables.

$n=1$

$$\int_{[c,d]=\varphi([a,b])} f(y)\,dy = \int f(\varphi(x))\varphi'(x)\,dx$$

$$y=\varphi(x)$$

$$dy=\varphi'(x)\,dx$$

where $\varphi : X \longrightarrow Y$

$[a,b] \longrightarrow [c,d]$

φ is bijective, C^1, $\varphi' \neq 0$

if φ is increasing then

$Y = [c,d] = [\varphi(a), \varphi(b)]$

if φ is decreasing then

$Y = [c,d] = [\varphi(b), \varphi(a)]$.

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \begin{cases} \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy & \text{if } \varphi \nearrow \\ \int_{\varphi(b)}^{\varphi(a)} f(y)\,dy & \text{if } \varphi \searrow . \end{cases}$$

$$= (\operatorname{sgn} \varphi') \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy .$$

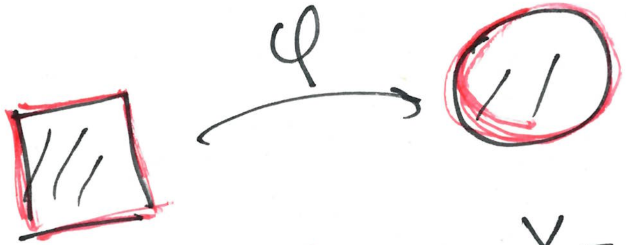
$$\Longrightarrow \boxed{\int_a^b f(\varphi(x))\,(\operatorname{sgn}\varphi')\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(y)\,dy.}$$

$$\boxed{\int_c^b f(\varphi(x))\,|\varphi'(x)|\,dx = \int_{\varphi(c)}^{\varphi(b)} f(y)\,dy}$$

In $\mathbb{R}^n$, suppose we have map

$$X \xrightarrow{\varphi} Y$$

bdd closed — bdd closed



$Y = Y_0 \cup C$

$X = X_0 \cup B$

Suppose $\varphi : X_0 \longrightarrow Y_0$ is C^1, bijective $\det J_\varphi(x) \neq 0$

$\forall x \in X_0$.

Let $Y = \varphi(X)$, $f : Y \to \mathbb{R}$ continuous function

/

$$\int_Y f(y)\,dy = \int_X f(\varphi(x))\,|\det J_\varphi(x)|\,dx$$

$n=1$ $\quad |\det J_\varphi(x)| = |\varphi'(x)|$.

Why do we have $|\det J_\varphi(x)|$? in the formula

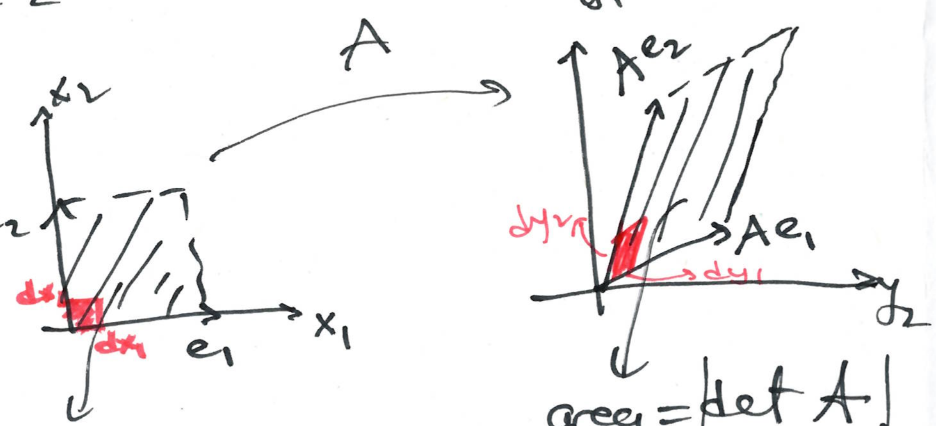
Ex. $\varphi : \mathbb{R}^n \to \mathbb{R}^n$

$$x \to Ax$$

A is $n \times n$ matrix.

$$\mathrm{Jac}_\varphi(x) = A.$$

$n = 2$



area $= |\det A|$

Volume = area 1.

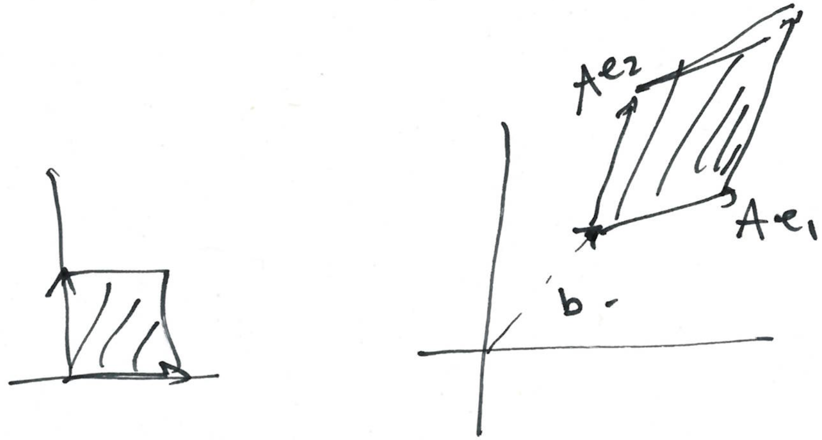
$$V(y_1, y_2) = \det A \, V(x_1, x_2)$$

$$dx_1 dx_2$$

$$dy_1 dy_2 = |\det A| \, dx_1 dx_2$$

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$

$$x \to b + Ax$$



$$\Delta V(y) = |\det A| \, \Delta V(x).$$

General $\varphi : \mathbb{R}^n \to \mathbb{R}^n$

non-linear, $x_0 \in \mathbb{R}^n$

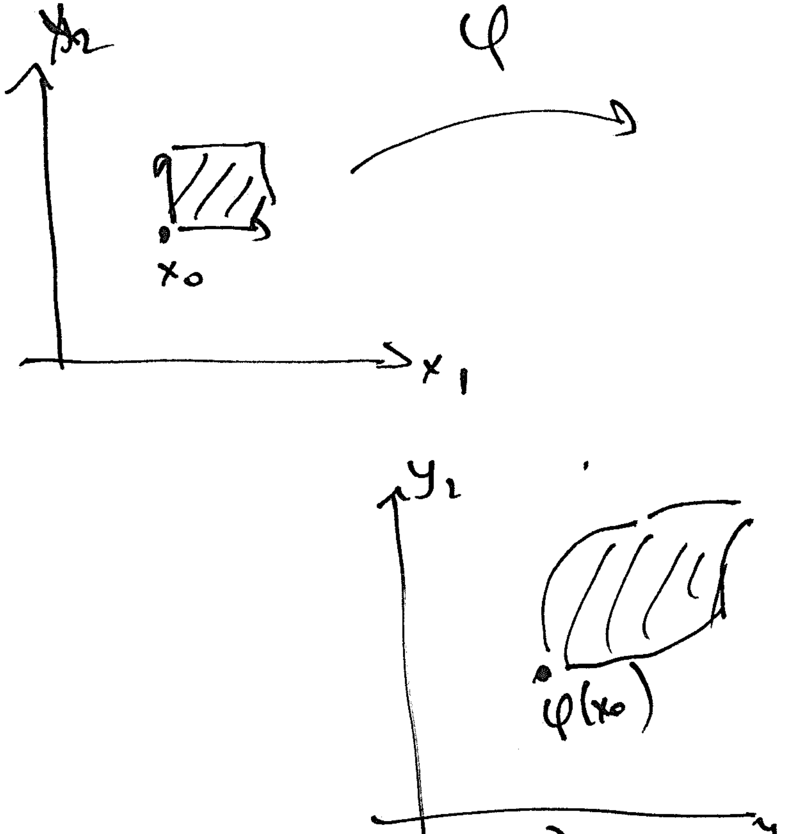
$d\varphi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$

linear map which is the differential.

If x is near x_0

$$\varphi(x) \approx \varphi(x_0) + J_\varphi(x_0)(x - x_0),$$

which is affine linear.



$$\Delta V(y_1, y_2) = |\det J_\varphi(x_0)| \, \Delta V(x_1, x_2)$$