15.12.20

Change of variables

$f: Y \subset \mathbb{R}^n \to \mathbb{R}$.

continuous function

Suppose we have a map

$\varphi: X \to Y$

where $X = X_0 \cup B$, $Y = Y_0 \cup C$

B, C boundary of X, Y resp.

Suppose $\varphi: X_0 \to Y_0$ is C^1

bijective, $\det J_\varphi(x) \neq 0, \forall x \in X_0$

Then we have the following change of variables formula

$$\boxed{\int_Y f(y)\,dy = \int_X f(\varphi(x))\,|\det J_\varphi(x)|\,dx}$$

note for $n=1$, if

$Y = [c,d]$, $X = [a,b]$

and $\varphi: X \to Y$

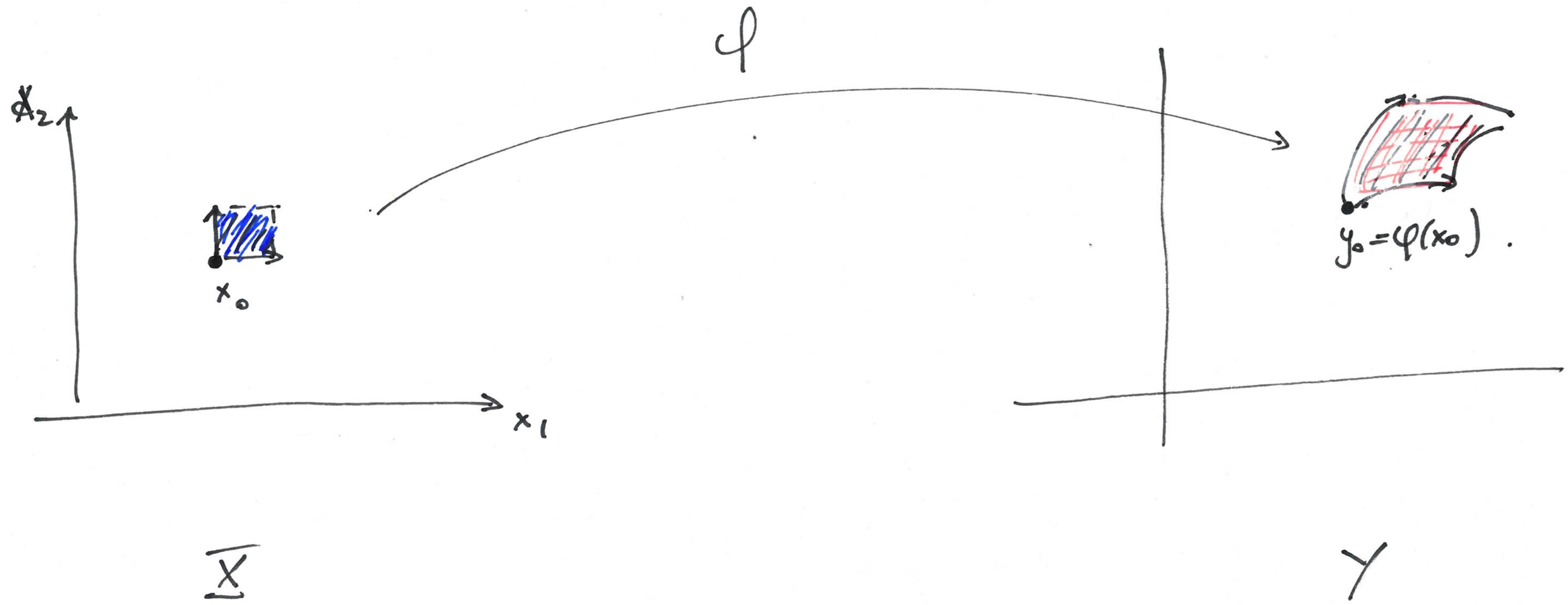
with $\varphi([a,b]) = [c,d]$

then we have

$$\boxed{\int_a^b f(\varphi(x))\,|\varphi'(x)|\,dx = \int_c^d f(y)\,dy.}$$

$|\varphi'(x)| = |\det J_\varphi(x)|$.

$$\varphi : X \longrightarrow Y$$



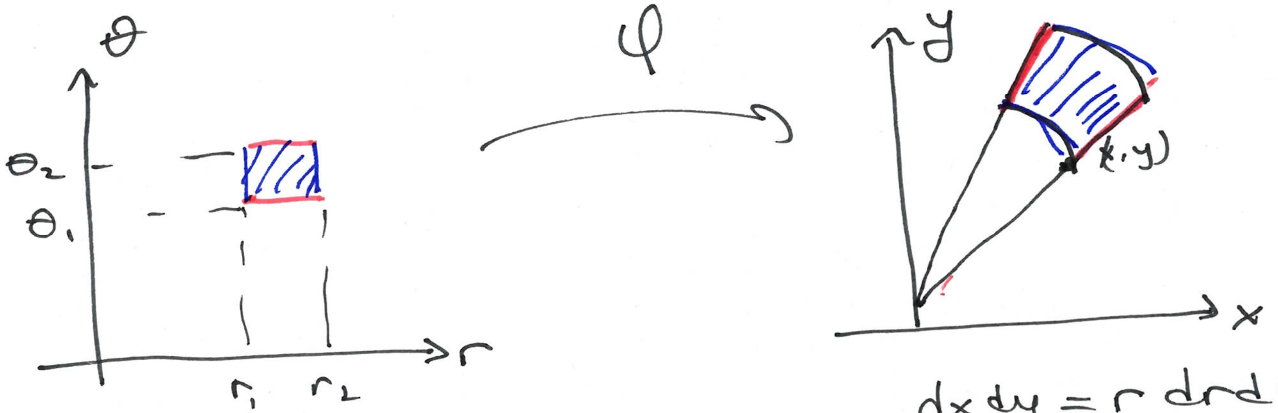
$$\Delta V(x_1, x_2) \longrightarrow \Delta V(y_1, y_2) = |\det J_\varphi(x_0)|\, \Delta V(x_1, x_2)$$

For x near x_0, $\varphi(x) \approx \varphi(x_0) + J_\varphi(x_0)(x - x_0)$:

ie φ looks like an affine map.

Important example

Polar coordinates



$$dx\,dy = r\,dr\,d\theta.$$

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

$$\varphi(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$$

$$|\det J_\varphi| = \left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{matrix}\right| = r$$

Ex: $\displaystyle\iint_D x\,dx\,dy$

$$\int_0^1\int_0^{\sqrt{1-x^2}} x\,dy\,dx$$

$$= \dots$$

$$\int_0^{\pi/2}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\int_0^1 r^2\,dr\right)\cos\theta\,d\theta$$

$$\underbrace{}_{=\left(\frac{r^3}{3}\Big|_0^1\right)}$$

$$= \frac{1}{3} \int_0^{\pi/2} \cos\theta \, d\theta = \frac{1}{3}.$$

(2) Find the volume between two paraboloids given by $z = x^2 + y^2$ and $z = 50 - x^2 - y^2$

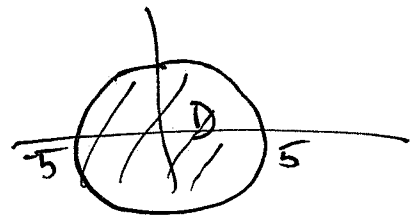
$$V = \iint_D \underbrace{(50 - x^2 - y^2) - (x^2 + y^2)}_{50 - 2(x^2+y^2)} \, dx\, dy.$$

We have to find the region of intersection : D

$$50 - x^2 - y^2 = x^2 + y^2$$

$$\Rightarrow 50 = 2x^2 + 2y^2$$

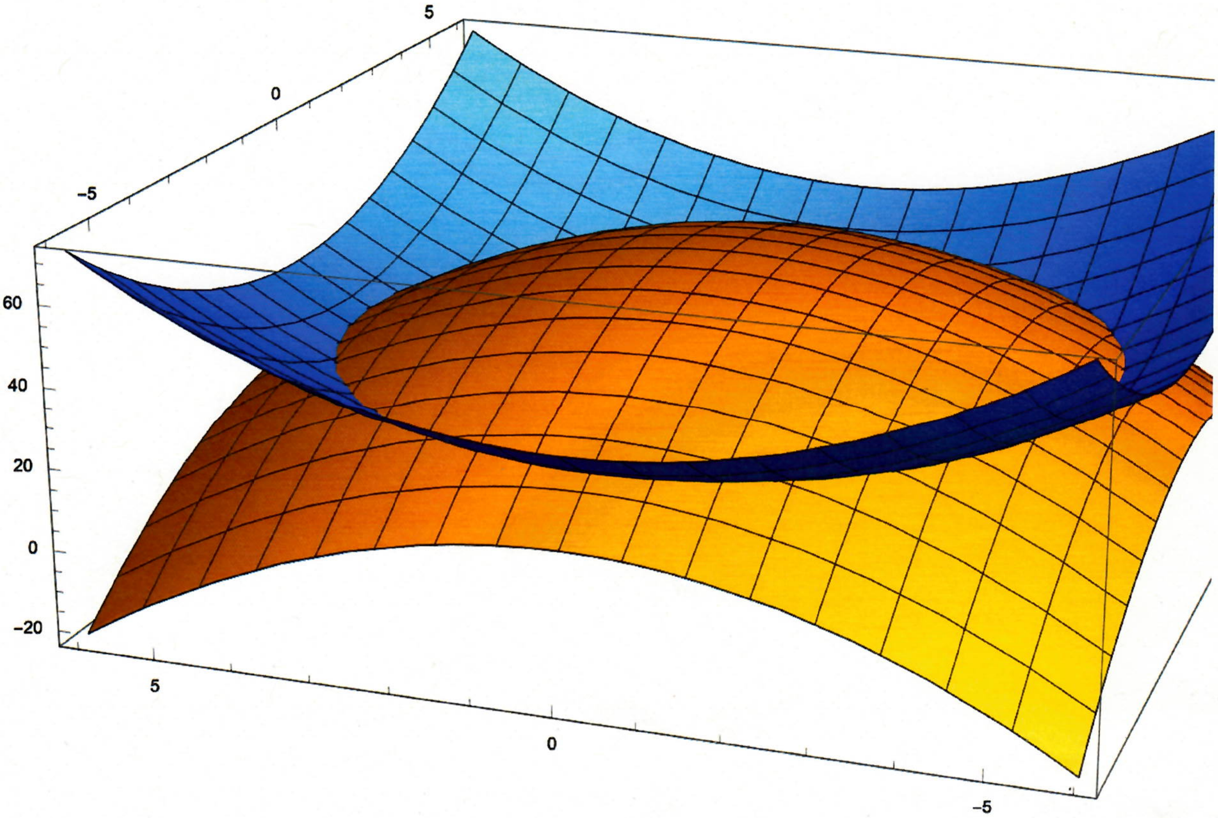
$$\Rightarrow x^2 + y^2 = 25$$



In polar coordinates

$$\int_0^{2\pi} \int_0^5 (50 - 2r^2) \, r\, dr\, d\theta = \cdots = 625\pi.$$

In[3]:= `Plot3D[{50 - x^2 - y^2, x^2 + y^2}, {x, -6, 6}, {y, -6, 6}]`

Out[3]= 

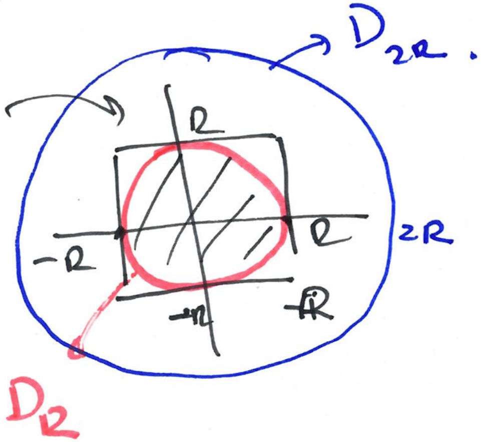
Ex:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= ?$$

$$\lim_{R\to\infty}\iint_{S_R} e^{-x^2-y^2}\,dx\,dy$$

where S_R



$$\iint_{D_R} e^{-x^2-y^2}\,dx\,dy$$

$$\int_0^R\int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr.$$

$$\int e^{-r^2} r\, dr \qquad r^2 = u$$

$$2r\,dr = du$$

$$\int e^{-u}\frac{du}{2} = -\frac{1}{2}e^{-u}$$

$$2\pi\int_0^R e^{-r^2} r\, dr = \frac{2\pi}{2}\left(1-e^{-R}.\right)$$

$$= \pi(1-e^{-R}).$$

$$\iint_{D_R} \cdots \le \iint_{S_R} e^{-x^2-y^2}\,dx\,dy \le \iint_{D_{2R}} \cdots$$

$$\begin{array}{ccc} \| & & \Downarrow \\ \pi(1-e^{-R}) & & \pi(1-e^{-2R}) \end{array}$$

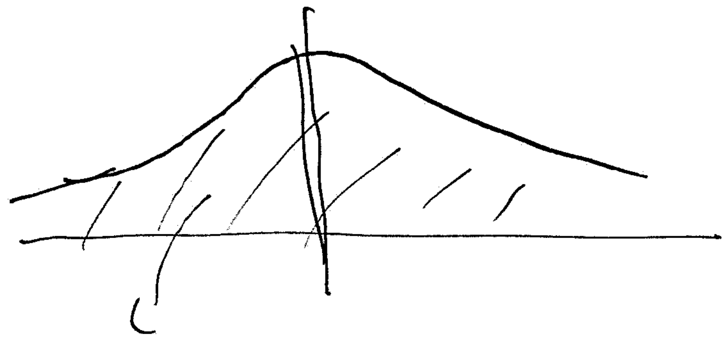
let $R \to \infty$.

$$\pi \le \lim_{R\to\infty} \iint_{S_R} e^{-x^2-y^2}\,dx\,dy \le \pi$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \pi.$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$$

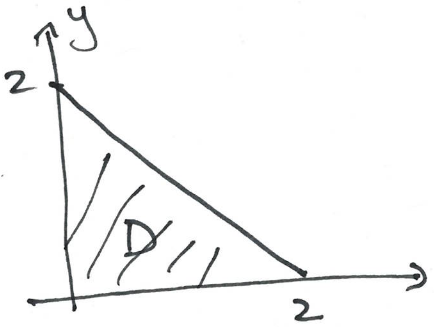
$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$



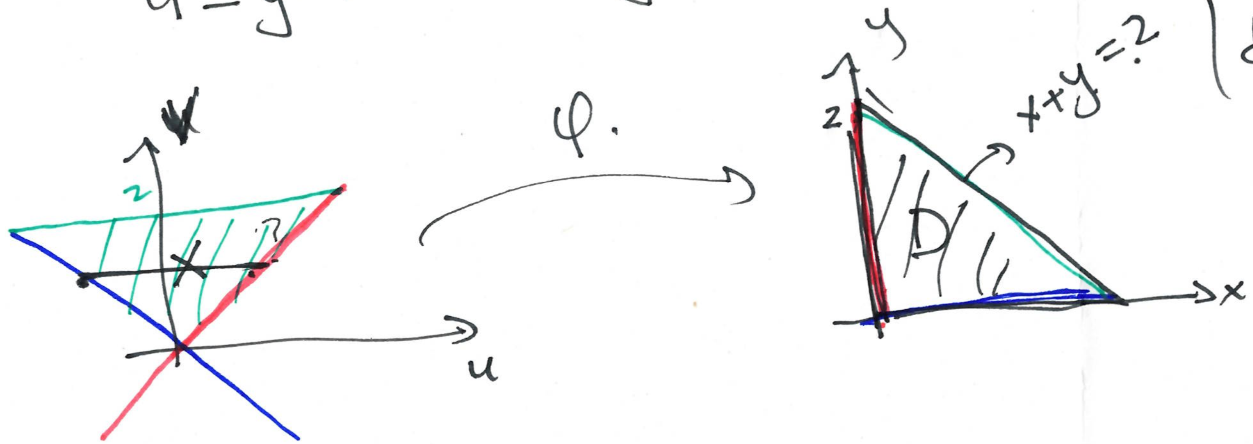
area $\sqrt{\pi}$.

Example:

$$\iint_D e^{(y-x)/(y+x)}\,dx\,dy$$



$u = y - x \qquad v = y + x$



$$\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$x = \frac{v-u}{2}, \quad y = \frac{u+v}{2}.$$

$$\varphi(u,v) = (x,y) = \left(\frac{v-u}{2}, \frac{u+v}{2}\right)$$

$$|\det J_\varphi| = \left|\det\begin{pmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}\right| = \frac{1}{2}.$$

$$dx\,dy = \frac{1}{2}\,du\,dv.$$

$x=0$ line , $x = \frac{v-u}{2}$

$\Rightarrow v=u$

$y=0$ line $y = \frac{u+v}{2} = 0$

$\Rightarrow u=-v.$

$x+y=2$ line $\Rightarrow v=2$

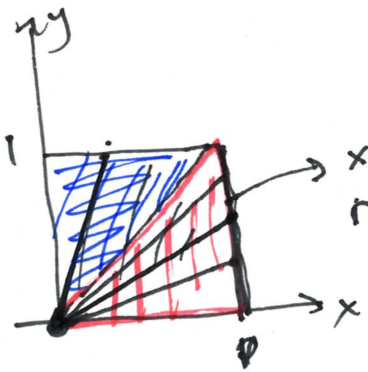
$$\iint_D e^{(y-x)/y+x} \, dx\,dy$$

$$= \iint_X e^{u/v} \frac{1}{2} \, du\,dv.$$

$$\frac{1}{2} \int_0^2 \int_{-v}^{v} e^{u/v} \, du\,dv. = \cdots$$

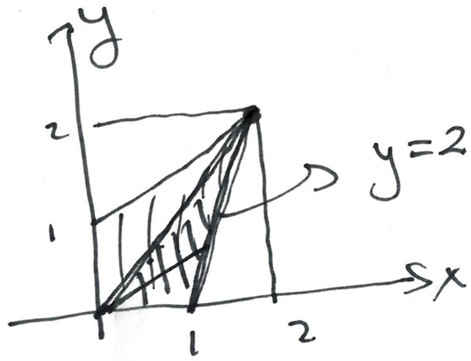
Clicker

$$\int_0^{\pi/4}\int_0^{1/\cos\theta} \dots \, dr\, d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} \dots \, dr\, d\theta$$



$x=1$

$r = \dfrac{1}{\cos\theta}$



$y = 2(x-1)$

$x = r\cos\theta$

$1 = r\cos\theta$

$r = \dfrac{1}{\cos\theta}$

$y=1$ line

$r\sin\theta = 1$

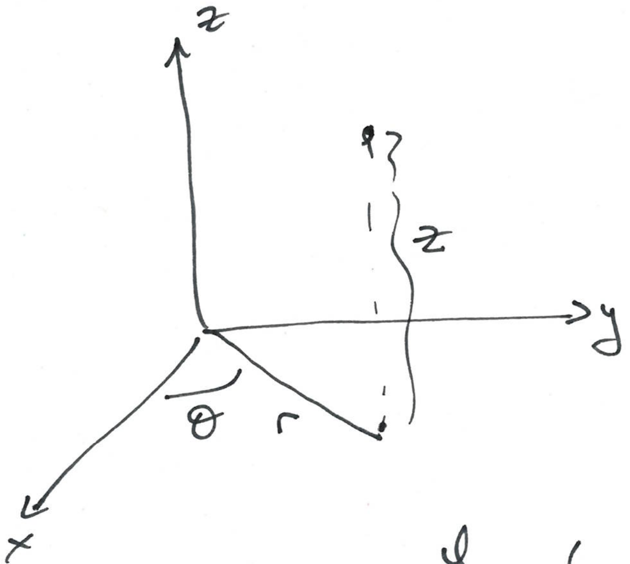
$r = \dfrac{1}{\sin\theta}$.

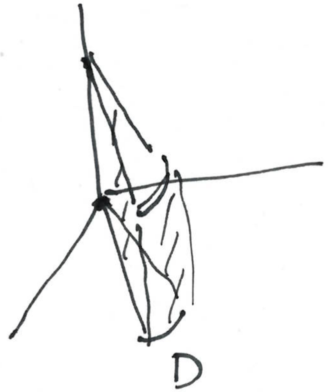
Exercise: Write this region in terms of polar coordinates

⋮

$\underline{n=3}$.

Cyclindrical coordinates.





$$0 \leq r \leq R$$

$$0 \leq z \leq H$$

$$\alpha \leq \theta \leq \beta$$

$$(r, \theta, z) \xmapsto{\varphi} (x, y, z).$$

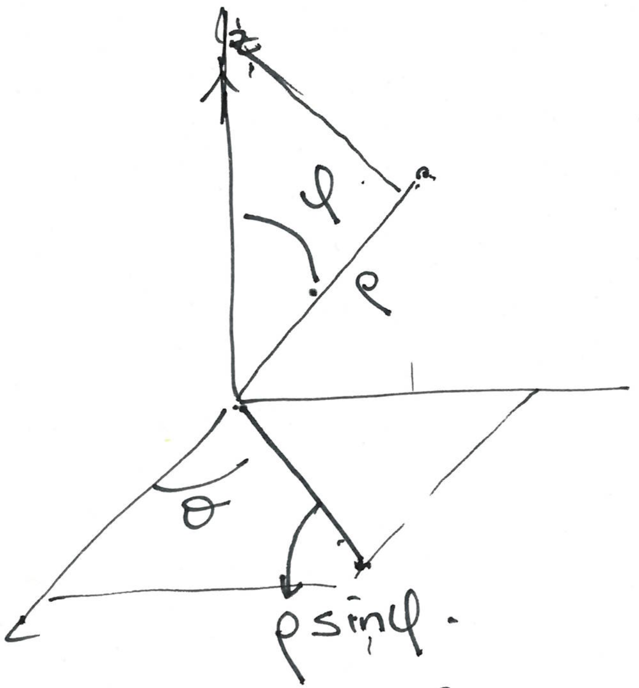
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\det |J_\varphi (r,\theta,z)| = r$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$\iiint_D \ldots dx\,dy\,dz = \int_0^R \int_0^H \int_\alpha^\beta \ldots r\,d\theta\,dz\,dr$$

Spherical coordinates.



$$\varphi : \overset{\rho}{[0,\infty)} \times \overset{\theta}{[0 \times 2\pi]} \times \overset{\varphi}{[0,\pi]} \longrightarrow \mathbb{R}^3$$

$$(\rho, \theta, \varphi) \to (x, y, z)$$

$$x = \rho \sin\varphi \cdot \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi .$$

$$|J_\varphi| = \rho^2 \sin\varphi$$

Volume of a sphere:

$$S : x^2 + y^2 + z^2 = r^2$$

$$\iiint_S 1 \, dx\, dy\, dz$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\underbrace{\int_0^{2\pi} d\theta}_{2\pi} \; \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{\underbrace{\cos\varphi \,\big|_0^{\pi}}_{2}} \; \underbrace{\int_0^{r} \rho^2 \, d\rho}_{\frac{r^3}{3}}$$

$$= \frac{4\pi r^3}{3}$$

/11.

§ 4.6 Green's thm.

let's recall fund. thm. of analysis in 1 variable.

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

where $F' = f$

$$\underbrace{\int_a^b F'(t)\,dt}_{\text{Integral over a region, } [a,b] \text{ of derivative of } F} = \underbrace{F(b) - F(a)}_{\text{values of } F \text{ at the boundary points}}.$$

Green's thm is simplest of a class of Theorems which relates integral of "some kind of derivative" of a function over a region, to the integral of the function over the boundary of the region.

The most common form of Green's thm

$$\iint_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$= \int_{\partial X} f \cdot ds$$

$f = (f_1, f_2)$

<u>We need some assumptions</u>:

1) The vector field $f = (f_1, f_2)$ is C^1 in the region X; so that $\operatorname{curl} f = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$ is integrable.

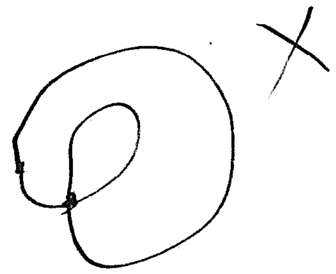
2) The region ~~X~~ is closed and bounded and its boundary is a simple closed curve if $\gamma : [a, b] \to \mathbb{R}^2$ is the curve which ~~goes~~ is the boundary of X

γ closed means : $\gamma(a)=\gamma(b)$

γ simple means that

$\exists$ no $s, t \in (a, b)$

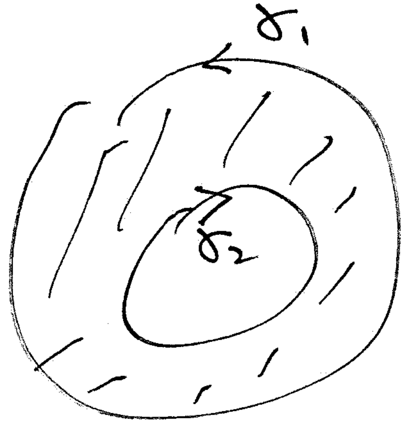
such that $\gamma(s)=\gamma(t)$



3) X is always on the left side of a tangent vector to the boundy.

We can allow union of simple curves as a boundary.

We again need that the region is to the left of the boundry.



** Green's theorem

Thm let $f : X \to \mathbb{R}^2$ C^1 vector-field, X closed and bounded where $\partial X = \bigcup_{i=1}^{n} \gamma_i$ union of simple closed curves so that X is always to the left of the curve $\gamma = \bigcup_{i=1}^{n} \gamma_i$ then

$$\iint_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \int_{\gamma} f\,ds$$

Rk. Green's thm can be used in both directions: ie

(1) We can use the double integral to calculate the line integral

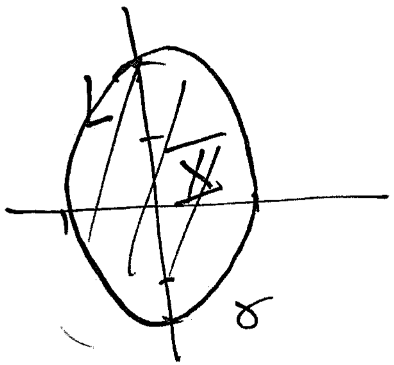
(2) We can use the line integ. to calculate a double integral.

Ex: $f = (y+3x,\ y-2x)$

X = region bdd by the ellipse $x^2 + \frac{y^2}{4} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ area is $\pi a b$

In special case $a=b$ this is a circle of area πa^2.



$$\int_\gamma f\, ds = \iint_X \left(\frac{\partial}{\partial x}(y-2x) - \frac{\partial}{\partial y}(y+3x) \right) dx\,dy .$$

$$= \iint_X -2-1\ dy\,dx = -3 \underbrace{\iint_X dx\,dy}_{\substack{\text{area of the ellipse} \\ 2\pi}} = -6\pi$$

or we can do directly the line integral.

$$\gamma : [0, 2\pi] \to \mathbb{R}^2$$
$$t \to (\cos t, 2\sin t)$$

$$\int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\int_0^{2\pi} (2\sin t + 3\cos t,\ 2\sin t - 2\cos t) \cdot (-\sin t,\ 2\cos t)\, dt$$

$$\vdots \quad \ddots \quad = -6\pi$$

In this example we used the double integral to calculate the line integral.

In general we can use Green's thm to calculate areas bounded by curves

$$\iint_X 1\, dx\, dy = \text{Area } X.$$

to calculate Area we look for a vector field $f = (f_1, f_2)$ so that $\operatorname{curl} f = 1$.

Many examples can be found:

e.g $f = (0, x)$

$$\text{curl} f = \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} 0 = 1$$

or $f = (-y, 0)$

$$\text{curl} f = 1.$$

or $f = (-y/2, \frac{x}{2})$

$$\text{curl} f = \frac{1}{2} + \frac{1}{2} = 1$$

Example Find the area enclosed by the curve

$$\gamma(t) = (t^2, t^3/3 - t)$$

$$-\sqrt{3} \leq t \leq \sqrt{3}$$

Area: Choose $f = (0, x)$

$$\int_\gamma f\, ds. = \int_{-\sqrt{3}}^{\sqrt{3}} f(\gamma(t)) \cdot \gamma'(t)\, dt$$

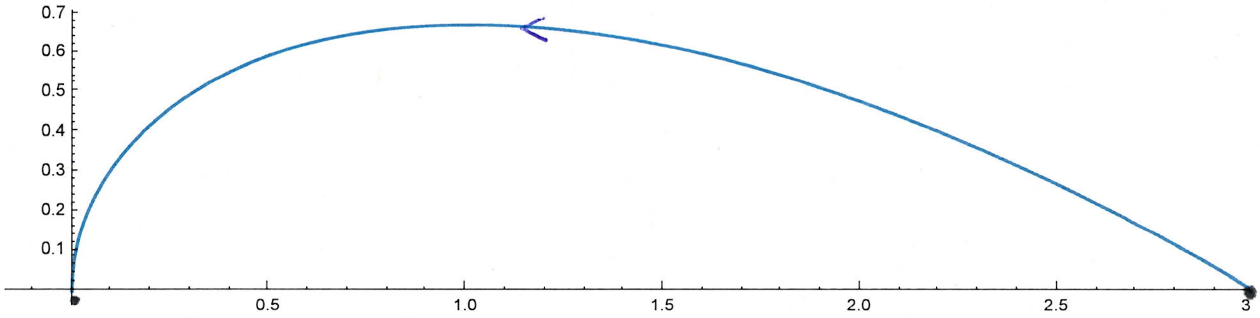
$$\int_{-\sqrt{3}}^{\sqrt{3}} (0, t^2) \cdot (2t, t^2 - 1)\, dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} t^4 - t^2\, dt = \left. \frac{t^5}{5} - \frac{t^3}{3} \right|_{-\sqrt{3}}^{\sqrt{3}}$$

$$\boxed{\frac{8}{5}\sqrt{3}}$$

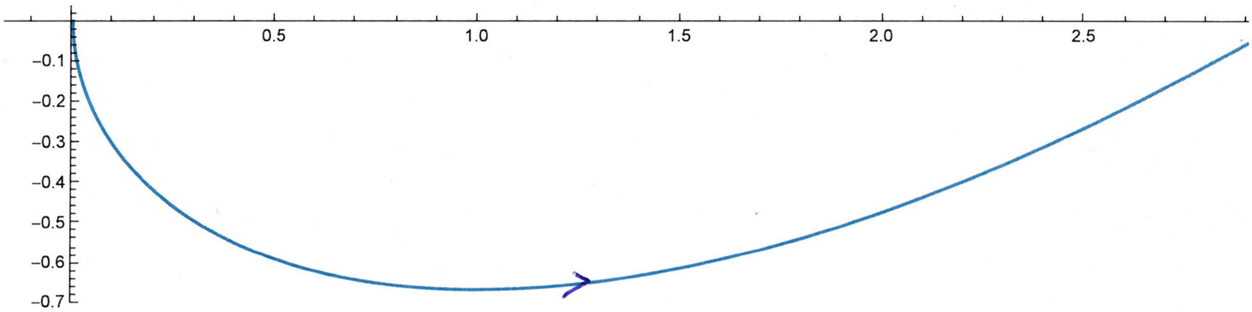
```
In[1]:= ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3], 0}]
```

Out[1]=



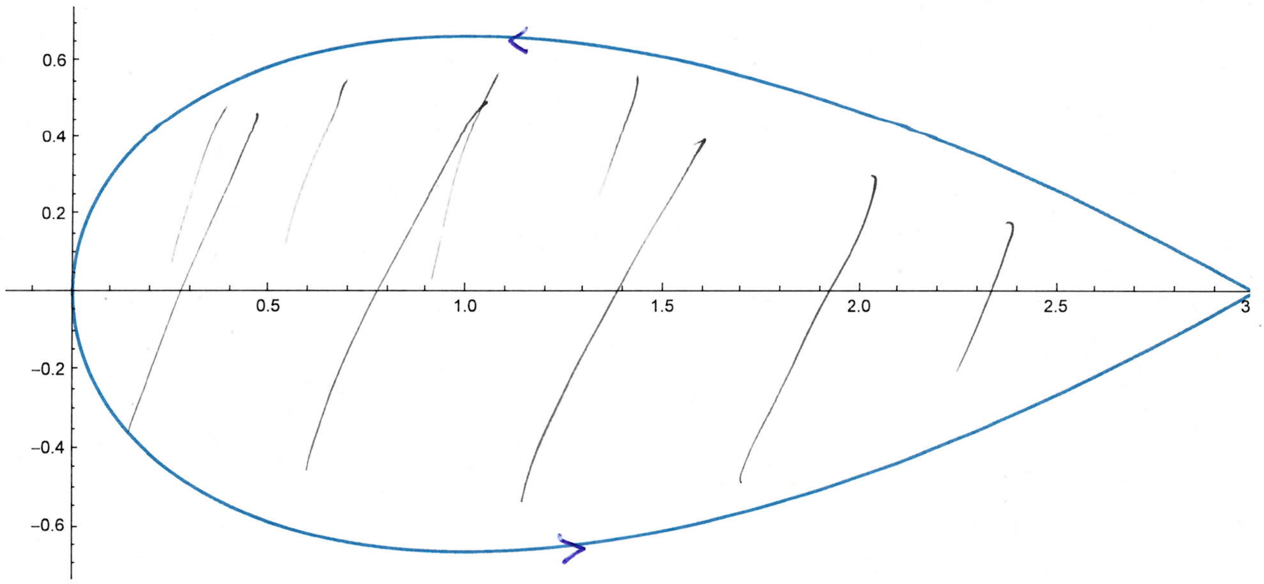

```
In[4]:= ParametricPlot[{t^2, t^3 / 3 - t}, {t, 0, Sqrt[3]}]
```

Out[4]=

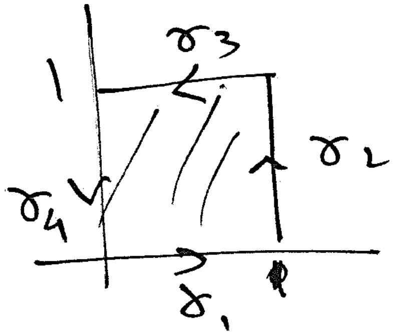



```
In[3]:= ParametricPlot[{t^2, t^3 / 3 - t}, {t, -Sqrt[3], Sqrt[3]}]
```

Out[3]=



Ex: let γ be the curve tracing the square with corners $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$



$f = (5 - xy - y^2,\ x^2 - 2xy)$

$$\int_\gamma f\,ds = ?$$

Greens thm: $\displaystyle\int_\gamma f\,ds = \int_0^1\int_0^1 \operatorname{curl} f\,dx\,dy.$

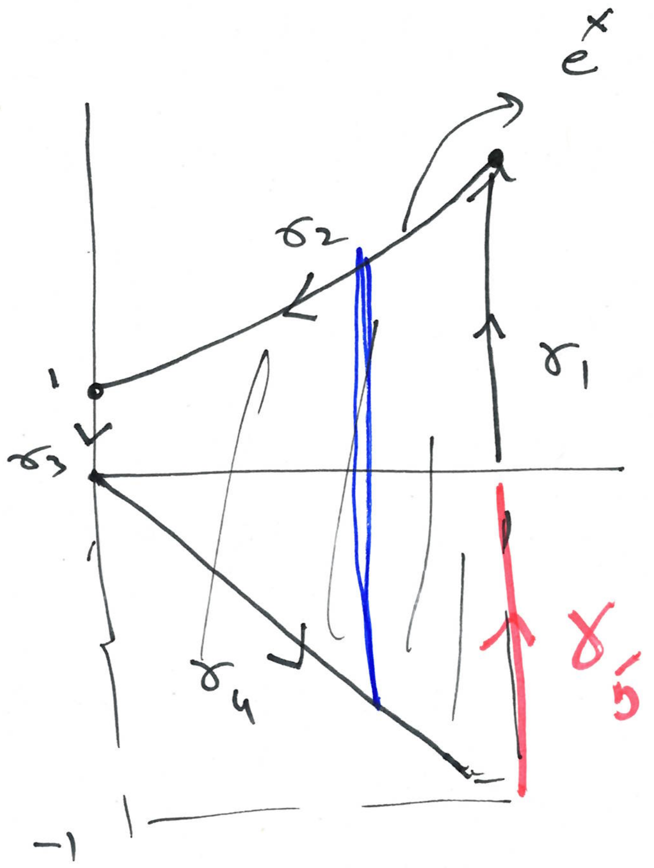
$$= \int_0^1\int_0^1 (2x-2y) - (-x-2y)\,dx\,dy. = \int_0^1\int_0^1 3x\,dx\,dy = \frac{3}{2}.$$

or

$$\int_\gamma f\,ds = \int_{\gamma_1} f\,ds + \int_{\gamma_2} f\,ds + \cdots \int_{\gamma_4} f\,ds$$

$$= \cdots \frac{3}{2}$$

__Ex__ Exam (2010)



$f = (xy^2, -y)$

$\gamma = \bigcup_{i=1}^{4} \gamma_i$

$$\int_\gamma f \, ds = ?$$

2 soln's.

1. $\gamma_1(t) = (1, t)$ $\quad 0 \le t \le e.$

$\gamma_2(t) = (t, e^t)$ $\quad t$ from 1 to 0.

$\gamma_3(t) = (0, t)$ $\quad t$ from 1 to 0.

$\gamma_4(t) = (t, -t)$ $\quad t$ from 0 to 1

$$\int_{\gamma_1} f\,ds = \int_0^e (t^2, -t)\cdot(0,1)\,dt$$

$$\int_{\gamma_2} f\,ds = \int_1^0 (te^{2t}, -e^t)\cdot(1, e^t)\,dt$$

$$\int_{\gamma_3} f\,ds = \int_1^0 (0, -t)\cdot(0,1)\,dt$$

$$\int_{\gamma_4} f\,ds = \int_0^1 (t^3, t)\cdot(1, -1)\,dt$$

$+$

$$= -\frac{e^2}{4} - \frac{1}{2}$$

OR We can use Green's thm by first closing the curve.

$$\int_{\gamma} f\,ds + \int_{\gamma_5} ds$$

$$= \iint_X \operatorname{curl} f \, dx\,dy.$$

$$= \iint_X (0 - 2xy)\,dx\,dy.$$

$$\cancel{= \int_0^1 \int (-2xy)}\;dx$$

$$\iint_X -2xy \, dx\, dy$$

$$= \int_0^1 \int_{-x}^{x} (-2xy)\, dy\, dx$$

$$= \cdots = -\frac{e^2}{4}$$

$\gamma_5(t) = (1, t)$ t from -1 to 0.

$$\int_{\gamma_5} f\, ds = \int_{-1}^{0} (t^2, -t) \cdot (0,1)\, dt$$

$$= \int_{-1}^{0} -t \, dt = +\frac{1}{2}$$

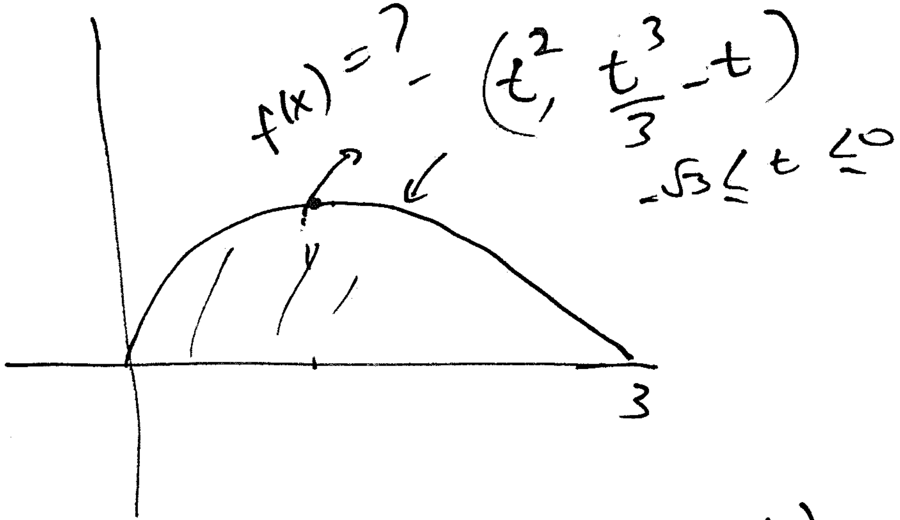
Hence

$$\int_\gamma f\, ds = \iint_X \operatorname{curl} f \, dx + \int_{\gamma_5} f\, ds.$$

$$= \frac{e^2}{4} - \frac{1}{2}$$

as before.

Example from p. 18 If you want to find the area using a double integral:



$$y = \frac{\left(-\sqrt{x}\right)^3}{3} + \sqrt{x}$$

$$= \sqrt{x}\left(1 - \frac{x}{3}\right)$$

$$\iint_X dx\,dy = \int_0^3 \int_0^{f(x)} 1\,dy\,dx$$

$$\int_0^3 \int_0^{\sqrt{x}(1-x/3)} dy\,dx$$

$$= \int_0^3 \sqrt{x}\left(1 - \frac{x}{3}\right) dx$$

$$= \frac{4\sqrt{3}}{5}$$

Upper part was for $t < 0$.

$x = t^2 \qquad y = \frac{t^3}{3} - t$

$t = -\sqrt{x}$ because $t < 0$

so that the whole area is $2 \cdot \frac{4\sqrt{3}}{5} = \frac{8\sqrt{3}}{5}$ as before.