D-MATH | Lineare Algebra I | HS 2022

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Serie 2

Exercises 2,3, and 6 are taken from Professor Einsiedler's Analysis lecture notes.
In this exercise sheet, we assume that $\mathbb{N}$, the set of natural numbers, denotes all integers greater or equal to 1.

1. **Logic.** Formulate the statement "there does not exist a largest natural number" and the statement "for ever natural number n there exists a strictly larger natural number" in First-order logic. Show by transformation in First-order logic the equivalence of both statements.

2. **Archimedean principle.** Describe the set

$$\bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} \mid -\frac{1}{n} \leqslant x \leqslant \frac{1}{n} \right\},$$

where n runs through the set $\mathbb{N}$ of natural numbers.

3. **Cartesian product.** Let X, Y be sets and A, A' subsets of X. Moreover, let B, B' be subsets of Y. Show

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Convince yourself, for example by drawing a picture, that there does not exist a similar formula for unions of sets.

4. **Maps and sets.** Let

$$f : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto ax + b \end{array}$$

for some non-zero $a \in \mathbb{R}$ and for some $b \in \mathbb{R}$. Draw the following set

$$\{(x, y) \in \mathbb{R}^2 \mid y \leqslant f(x)\} \cap \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0\} \cap \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$$

   (a) for $a = 2$, $b = 1$;
   (b) for $a = -2$, $b = 1$.

5. **Inverses.**

   (a) Let $f : X \to Y$ be a map. Assume that there exist maps $g_1 : Y \to X$ and $g_2 : Y \to X$ such that

$$g_1 \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g_2 = \mathrm{id}_Y .$$

   Show then that $g_1 = g_2$. What can you say about f?

(b) Give an example of a function that admits a right-inverse but is not bijective. By right-inverse, we mean that letting f be a map from X to Y, there exists a map g from Y to X such that $f \circ g = \mathrm{id}_Y$.

6. **Maps and operations on sets.** Consider a function $f : X \to Y$. Let $A, A' \subseteq X$ and $B, B' \subseteq Y$ be subsets of X and Y respectively.

   (a) Prove that $f(f^{-1}(B)) \subseteq B$ is true. Under what conditions for f is equality guaranteed?

   (b) Prove that $f^{-1}(f(A)) \supseteq A$ is true. Under what conditions for f is equality guaranteed?

   (c) Show the equalities

   $$f(A \cup A') = f(A) \cup f(A'), \qquad f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B').$$

   (d) Prove that $f(A \cap A') \subseteq f(A) \cap f(A')$ and that equality is satisfied, if f is injective. Verify in that case that also $f(A \setminus A') = f(A) \setminus f(A')$ is true.

   (e) Prove that $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$ and $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ are true.

In summary you should remember that forming the preimmage commutes with all the set theoretic operations discussed in the lecture (including union, intersection, complement), while forming the image only satisfies this for unions or under more restrictive conditions on f.