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Serie 3

1. **Linear system.** Using Gauss elimination, find all the solutions to the following system of linear equations over $\mathbb{R}$:

$$\begin{cases} x + 2y + 2z + w &= -1 \\ 3x + 6y + 2z + 5w &= 1 \end{cases}$$

2. **Fields.** Consider the field $\mathbb{F}_5 := \mathbb{Z}/5\mathbb{Z}$. Its elements are $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}$, where $\overline{n}$ denotes the residue class of n modulo $5\mathbb{Z}$. Calculate:

   (a) all pairs (x, y) satisfying $x + y = \overline{0}$;

   (b) the elements $\frac{\overline{3}}{\overline{4}} + \frac{\overline{1}}{\overline{3}}$ in terms of $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}$;

   (c) the value of $\overline{4}^{2022}$.

3. **Fields.** Prove that for any $a, b \in \mathbb{F}_3$,

$$(a + b)^3 = a^3 + b^3.$$

4. **Linear System.** Fix $b_1, b_2, b_3 \in \mathbb{R}$. Determine when the following linear system of equations has a solution and describe its set of solutions $S \subseteq \mathbb{R}^3$ when it does

$$\begin{pmatrix} 0 & 7 & -2 \\ -1 & 2 & -1/2 \\ 4 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

5. **Fields.** Let $(k, +, \cdot, 0, 1)$ be a field and let $\alpha \in k$ such that $x^2 = \alpha$ does not have any solutions in k. Let τ be a formal symbol outside of k such that $\tau^2 = \alpha \in k$. Show that

$$k[\tau] = \{a + b\tau \mid a, b \in k\}$$

with the following operations

$$\begin{array}{rccc} + : & k[\tau] \times k[\tau] & \to & k[\tau] \\ & (a + b\tau, c + d\tau) & \mapsto & a + c + (b + d)\tau \end{array}$$

$$\begin{array}{rccc} \cdot : & k[\tau] \times k[\tau] & \to & k[\tau] \\ & (a + b\tau, c + d\tau) & \mapsto & ac + \alpha bd + (bc + ad)\tau \end{array}$$

as its addition and multiplication, and equipped with $0 + 0\tau$ as its 0 and $1 + 0\tau$ as its 1 is a field.

Can you give explicit examples of this construction?

6. **Fields.** Let k be a finite field.

(a) Let S be the sum of all elements of k. Show that $S = 0$ is satisfied if and only if k has more than two elements.
Hint: What are the properties of the map

$$\begin{array}{rccc} m_b : & k & \to & k \\ & x & \mapsto & b \cdot x \end{array}$$

for $b \in k^* = k \setminus \{0\}$?

(b) Let $M = \prod_{x \in k^*} x$ be the product of all non-zero Elements of k. Show that $M = -1$.
Hint: Consider the map $k^* \ni x \mapsto \frac{1}{x} \in k^*$.