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Serie 4

1. Let $m \in \mathbb{R}$. Describe the solutions of the following system of equations depending on m:
$$\begin{cases} x + my &= -3 \\ mx + 4y &= 6 \end{cases}$$
When is the set of solutions S a linear subspace of $\mathbb{R}^2$? Give a geometrical interpretation of S depending on m.

2. Which of the following sets are linear subspaces of the given vector spaces? What changes when $\mathbb{R}$ is replaced by $\mathbb{F}_2$ in (b) and (c)?
   (a) $S_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 2x_3\} \subseteq \mathbb{R}^3$
   (b) $S_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^4 = 0\} \subseteq \mathbb{R}^2$
   (c) $S_3 := \{(\mu + \lambda, \lambda^2) \in \mathbb{R}^2 \mid \mu, \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^2$

3. Let K be a field in which $1 + 1 \neq 0$ and consider the space
$$V = K^K = \mathrm{Abb}(K, K) := \{f : K \to K\}.$$
Recall from the lectures that it is a vector space when endowed with scalar multiplication, namely $(\alpha \cdot f)(x) = \alpha f(x)$, $\forall \alpha \in K$, $\forall x \in K$, and with point wise addition, i.e. $(f + g)(x) = f(x) + g(x)$, $\forall x \in K$.
Now let
$$V_{even} := \{f : K \to K \mid f(-x) = f(x)\, \forall x \in K\},$$
$$V_{odd} := \{f : K \to K \mid f(-x) = -f(x)\, \forall x \in K\}.$$
Show that V_{even} and V_{odd} are linear subspaces of V, that
$$V_{even} + V_{odd} := \{v + w \mid v \in V_{even},\, w \in V_{odd}\} = V$$
and that $V_{even} \cap V_{odd} = \{0\}$.

4. Let ∞ and $-\infty$ denote 2 distinct objects, neither of which is in $\mathbb{R}$, Define an addition and a scalar multiplication on $V := \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as follows: in $\mathbb{R}$, addition and multiplication are defined as usual. For $t \in \mathbb{R}$ define
$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty.$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = (-\infty), \quad \infty + (-\infty) = (-\infty) + \infty = 0.$$

Is V a vector space over $\mathbb{R}$?

5. Let X be a set and let P be its power set (this means that P is the set of all subsets of X). For all $A, B \in P$ and for $\lambda \in \mathbb{F}_2$, define

$$A \triangle B := (A \cup B) \setminus (A \cap B)$$

$$\lambda \cdot A := \begin{cases} \varnothing, & \text{for } \lambda = 0, \\ A, & \text{for } \lambda = 1. \end{cases}$$

Show that $(P, \triangle, \cdot, \varnothing)$ is a $\mathbb{F}_2$-vector space.

6. Let V be a K-vector space and let $V_1, V_2.V_3 \subseteq V$ be linear subspaces, none of which is contained in another. Determine with proof if $V_1 \cup V_2 \cup V_3$ is always, sometimes or never a linear subspace of V.

   Hint: Try different fields K to obtain examples.

Multiple Choice questions. Each question can admit several answers.

Question 1. Which of the following sets are linear subspaces of the given vector spaces?

- $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 + 5x_2 + 3x_3 = 0, 2x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$
  You saw in the lectures that sets of solutions of systems of homogeneous linear equations are linear subspaces.

- $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3\} \subseteq \mathbb{R}^3$
  It is not a linear subspace since $(0, 0, 0)$ is not in it.

- $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\} \subseteq \mathbb{R}^2$
  It is not a linear subspace since multiplying any element by a negative scalar λ reverses the inequality. Hence for any (x_1, x_2) in the set, $\lambda(x_1, x_2)$ is not in the set anymore.

- $\{(0, x, 2x, 3x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^4$
  It is a linear subspace since for any $v = (0, x, 2x, 3x)$, $w = (0, y, 2y, 3y)$ in the set and any $\lambda \in \mathbb{R}$, we have

  $$v + \lambda w = (0, x + \lambda y, 2x + 2\lambda y, 3x + 3\lambda y) = (0, z, 2z, 3z),$$

  for $z = x + \lambda y$. Hence $v + \lambda w$ stays in the set.

- ○ $\{(x^4, x^3, x^2, x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^4$
  It is not a linear subspace. Note that $v := (1, 1, 1, 1)$ lies in this set but that $2v$ does not since there is no $x \in \mathbb{R}$ such that $2 = x^4 = x^3 = x^2 = x$.

Question 2. Consider the set of pairs of positive real numbers $\mathbb{R}^2_+$. The addition on $\mathbb{R}^2_+$ is defined as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}.$$

We now consider three different definitions of scalar multiplication, for a $\lambda \in \mathbb{R}$:

- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$
- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} e^\lambda x_1 \\ e^\lambda x_2 \end{pmatrix}$
- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1^\lambda \\ x_2^\lambda \end{pmatrix}$

According to which definition of scalar multiplication does $\mathbb{R}^2_+$ with the addition defined above become a $\mathbb{R}$-vector space?

- ○ First definition
  No. If $\lambda < 0$ then $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \notin \mathbb{R}^2_+$
- ○ Second definition
  No. Distributivity $\lambda(v + w) = \lambda v + \lambda w$ is not verified.
- ○ Third definition
  It checks all of the axioms.