D-MATH | Lineare Algebra I | HS 2022
Prof. M. Einsiedler
Prof. P. Biran

Serie 5

1. **Polynomials.** Consider the polynomials

$$p_1(x) = x^3 + x^2$$
$$p_2(x) = x^2 - 2x - 4$$
$$p_3(x) = 3x + 4$$
$$p_4(x) = 2x + 3$$

   (a) Express the polynomial $2x^3 + 3x^2 - 1$ as a linear combination of the p_i, $i = 1, 2, 3, 4$.

   (b) Calculate the linear span $\mathrm{Sp}(p_1, p_2, p_3, p_4)$.

2. **Dimension 2.** Let $v \in \mathbb{R}^2 \setminus \{(0,0)\}$ and let $w \in \mathbb{R}^2 \setminus \{(0,0)\}$ be such that $w \neq \alpha v$ for all $\alpha \in \mathbb{R}$.

   (a) Show that $\mathrm{Sp}(v, w) = \mathbb{R}^2$.

   (b) Show that the only subspaces of $\mathbb{R}^2$ are $\{(0,0)\}$, $\mathrm{Sp}(v)$ for all $v \in \mathbb{R}^2$, and $\mathbb{R}^2$.

3. **Subvectorspaces.** Do the operations $+$ and $\cap$ on subvectorspaces satisfy distributivity? In other words, do the following equations hold for all linear subspaces?

$$U \cap (V_1 + V_2) = (U \cap V_1) + (U \cap V_2)$$
$$U + (V_1 \cap V_2) = (U + V_1) \cap (U + V_2)$$

   If not, is at least one inclusion satisfied?

   Recall that for two subspaces V_1 and V_2 of a vector space V,

$$V_1 + V_2 = \{u + v \mid u \in V_1,\ v \in V_2\}.$$

4. **Subspaces and equations.** Let K be a field. Fix $x \in K^n$ and $b \in K^m$. Define

$$U := \{A \in M_{m \times n}(K) \mid A \cdot x = b\}.$$

   For which values of x and b is $U \subseteq M_{m \times n}(K)$ a linear subspace?

5. **Polynomials.** Prove that $K[x]$ is *not* finite-dimensional over K.

6. **Sequences.**

(a) Let K_0^∞ be the set of finitely-supported sequences, i.e.

$$K_0^\infty = \{(a_0, a_1, a_2, \dots, a_n, \dots) \mid \forall i \in \mathbb{N}\, a_i \in K \wedge \exists N \geqslant 0 \text{ s.t. } a_n = 0\, \forall n \geqslant N\}.$$

Write the smallest (with respect to inclusion) generating subset $E \subsetneq K_0^\infty$ that you can think of and justify your answer.

(b) Do the same for

$$K_{cst}^\infty := \{(a_0, a_1, a_2, \dots, a_n, \dots) \mid \forall i \in \mathbb{N}\, a_i \in K \wedge \exists c \in K \exists N \geqslant 0 \,:\, a_n = c\, \forall n \geqslant N\},$$

the set of eventually constant sequences.