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Serie 6

1. For each of the following sets, prove whether or not they are linearly independent over $\mathbb{R}$:

$$S_1 = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-3\\9\end{pmatrix}, \begin{pmatrix}1\\3\\-4\\7\end{pmatrix}, \begin{pmatrix}2\\0\\1\\3\end{pmatrix} \right\},$$

$$S_2 = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \right\}.$$

2. Are the following sets linearly independent over $\mathbb{R}$?
   (a) $\{(1,0,0),(0,2,t),(2,4,t^2)\}$ for t in $\mathbb{R}$;
   (b) The set of columns of an upper triangular matrix $A \in M_{n\times n}(\mathbb{R})$ with $A_{ii} \neq 0$ for all $1 \leqslant i \leqslant n$. We define an upper triangular matrix to be a matrix whose entries under the diagonal all vanish, i.e. a matrix $A = (A_{ij})_{1\leqslant i,j\leqslant n}$ such that $A_{ij} = 0$ whenever $j < i$.
   (c) $\{f, g\} \subseteq \mathrm{Abb}(\mathbb{R},\mathbb{R})$, where $f(x) = \sin(x)$ and $g(x) = \cos(x)$;
   (d) $\{f, g\} \subseteq \mathrm{Abb}(\mathbb{R},\mathbb{R})$, where $f(x) = e^{rx}$ and $g(x) = e^{sx}$, for fixed $s, r \in \mathbb{R}$.

3. Consider $A_1, A_2 \in M_{2\times 3}(\mathbb{R})$, given by

$$A_1 = \begin{pmatrix}1 & 2 & 3\\0 & 0 & 1\end{pmatrix}, \quad A_2 = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

   (a) Show that $\{A_1, A_2\}$ is linearly independent over $\mathbb{R}$.
   (b) Let

$$M := \left\{ \begin{pmatrix}a & b & c\\d & e & f\end{pmatrix} \in M_{2\times 3}(\mathbb{R}) \mid d = e = 0, b - a = f, 3a = c \right\}$$

   Prove that $\mathrm{Sp}(A_1, A_2) = M$.
   (c) Find $A_3 \in M_{2\times 3}(\mathbb{R})$ such that $\{A_1, A_2, A_3\}$ is linearly independent. Is $\mathrm{Sp}(A_1, A_2, A_3) = M_{2\times 3}(\mathbb{R})$ for any choice of such an A_3?

4. Show that
$$U = \{f \in \mathrm{Abb}(\mathbb{F}_5, \mathbb{F}_5) \mid \sum_{i=0}^{4} f(\bar{i}) = 0\} \subseteq \mathrm{Abb}(\mathbb{F}_5, \mathbb{F}_5)$$
is a linear subspace. Determine a basis of U.

5. Let V be a vector space over some field K that admits a countable basis. Show that every linearly independent subset $S \subseteq V$ is finite or countable.

6. Prove that the functions
$$\varphi_a : \mathbb{R}_{>0} \to \mathbb{R}, \quad x \mapsto \frac{1}{a+x}$$
for all $a \in \mathbb{R}_{\geqslant 0}$ are linear independent.

   Hint: Use that a non-zero polynomial only has finitely many zeros.

Multiple Choice questions. Each question can admit several answers.

Question 1. Let V be a vector space over K. Which of the following assertions is true ?

- Let $v \in V$, then the set
$$W := \{w \in V \mid \exists \lambda \in K : w = \lambda v\}$$
is a linear subspace of V.
- A subset $W \subset V$ is a linear subspace if and only if $\mathrm{Sp}(W) = W$.
- Let $S_1, S_2 \subset V$ be subsets. Then $\mathrm{Sp}(S_1 \cup S_2) = \mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$.
- Let $S_1, S_2 \subset V$ be subsets. Then $\mathrm{Sp}(S_1 \cap S_2) \subseteq \mathrm{Sp}(S_1) \cap \mathrm{Sp}(S_2)$.

Question 2. Let V be a vector space and let $S_1, S_2 \subseteq V$ with $S_1 \subsetneq S_2$. Which of the following are true?

(a) If S_1 is a linearly independent set, when is S_2 a linearly independent set?
   - Always
   - Never
   - Sometimes

   If now S_2 is a linearly independent set, when is S_1 a linearly independent set?
   - Always
   - Never
   - Sometimes

(b) Answer the previous question, replacing "linearly independent set" with "generating set for V".

(c) Answer question (a), replacing "linearly independent set" with "basis for V".