Serie 7

1. Compute the dimension and find a basis of the following spaces

   (a) The space of upper triangular matrices in $M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$ (for a definition, see Serie 6, exercise 2.b));

   (b) The space of diagonal matrices in $M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$, where diagonal matrices are all matrices $A = (a_{ij})$ such that $a_{ij} = 0$ whenever $i \neq j$;

   (c) The space of symmetric matrices

   $$W = \left\{A \in M_{n\times n}(\mathbb{R}) \mid A^T = A\right\},$$

   where $\cdot^T$ denotes the operation $A = (a_{ij})_{1\leqslant i,j\leqslant n} \mapsto (a_{ji})_{1\leqslant i,j\leqslant n}$;

   (d) The space of matrices $A \in M_{n\times n}(\mathbb{F}_2)$ such that the sum of the columns of A is the null vector.

2. Let W be the linear subspace generated by the vectors

   $$v_1 = \begin{pmatrix}1\\2\\1\\2\end{pmatrix},\ v_2 = \begin{pmatrix}0\\0\\-1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}1\\4\\2\\3\end{pmatrix},\ v_4 = \begin{pmatrix}0\\2\\1\\1\end{pmatrix}.$$

   (a) Determine a system of equations whose solution is W.

   (b) Find all the possible bases of W that can be built using v_1, v_2, v_3, v_4. How many are there?

3. Determine a basis and the dimension of the following spaces:

   (a) The set of solutions $S \subseteq \mathbb{R}^3$ of

   $$\begin{aligned} x + y - z &= 0 \\ 3x + y + 2z &= 0 \\ 2x + 3z &= 0 \end{aligned}$$

   (b) $\{0\}$;

   (c) $\{(z, w) \in \mathbb{C}^2 \mid z + iw = 0\}$ as a vector space over $\mathbb{C}$;

   (d) $\{(z, w) \in \mathbb{C}^2 \mid z + iw = 0\}$ as a vector space over $\mathbb{R}$.

4. Let K be a field, fix $g(X) := X + 5 \in K[X]$, and let $d \geqslant 1$. Compute the dimension of the space

$$W = \{h \in K[X] \mid \deg(h) \leqslant d \wedge \exists f \in K[X] : h = gf\}.$$

5. Consider the following subspace of K^n :

$$U := \left\{ (\alpha_1, \ldots, \alpha_n) \in K^n \mid \sum_{i=1}^{n} \alpha_i = 0 \right\},$$
$$D := \{(\alpha, \ldots, \alpha) \in K^n \mid \alpha \in K\}.$$

Determine a basis and compute the dimension of $U, D, U \cap D$, and $U + D$.

Remark: Do not forget to consider the case where K is a field such that $n \cdot 1 = 0$.

6. Determine a basis and compute the dimension of the space

$$\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \wedge (f + f'' = 0)\}.$$

Hint: Note that f, f', and f'' exist and are continuous. Moreover, you can use the following result from Analysis: the function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} : f(x) = 0$ is the unique continuous solution to

$$\begin{cases} f + f'' = 0 \\ f(0) = f'(0) = 0 \end{cases}$$