D-MATH | Lineare Algebra I | HS 2022
Prof. M. Einsiedler
Prof. P. Biran

Serie 8

1. Let F be a field. Find a linear complement of the following subspaces in $M_{n\times n}(F)$ (see Serie 7 for definitions):
   (a) The subspace of upper triangular matrices;
   (b) The subspace of symmetric matrices.

2. Let $b, c \in \mathbb{R}$ and let $\mathbb{R}[x]$ denote the space of polynomial functions of 1 variable with coefficients in $\mathbb{R}$. Define $T : \mathbb{R}[x] \to \mathbb{R}^2$ by
$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + cp(0)^2\right).$$
Show that T is linear if and only if $b = c = 0$.

3. Suppose that U and V are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exists 2 vectors in $U \cap V$ such that neither of these vectors is a scalar multiple of the other.

4. Suppose that $\{v_1, \ldots, v_m\}$ is linearly independent in V and let $w \in V$. Prove that
$$\dim \operatorname{Sp}(v_1 + w, v_2 + w, \ldots, v_m + w) \geqslant m - 1.$$

5. Let V be a vector space over a field F and consider 3 linear subspaces U_1, U_2, U_3 such that $V = U_1 + U_2 + U_3$ and for $i, j \in \{1, 2, 3\}$ such that $i \neq j$, we have $U_i \cap U_j = \{0\}$.
Is it true that for any $v \in U_1 + U_2 + U_3$ there exists a unique triple (u_1, u_2, u_3) with $u_i \in U_i$ such that $v = u_1 + u_2 + u_3$?

6. Let V be a finite-dimensional vector space over a field F and let
$$V \supseteq U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots$$
be an infinite sequence of nested subspaces.
   (a) Show that this sequence stabilises, i.e. show that there exists a $N \in \mathbb{N}$ such that for all $n \geqslant N : U_n = U_N$.
   (b) Is this still the case if we now assume that V is infinite-dimensional?
   (c) Assume that V is infinite-dimensional and that $\dim U_n \geqslant 1$ for all $n \in \mathbb{N}$. What can you say about $\bigcap_{n\in\mathbb{N}} U_n$?

Multiple Choice Fragen. More than one answer can be correct.

Frage 1. Which of the following maps are linar?

- $f : \mathbb{R}^2 \to \mathbb{R}^3,\ (x, y) \mapsto (x + y, 2x, 0)$
- $f : \mathbb{R}^2 \to \mathbb{R}^3,\ (x, y) \mapsto (x + y, 2x, 0)$
- $f : K^3 \to K^2,\ (x, y, z) \mapsto (\alpha x + \beta y + \gamma z, \delta x + \varepsilon y + \eta z)$ for fixed $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ in the field K

Frage 2. Which of the linear maps below can be written as $x \mapsto Ax$, where A is the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- $f : \mathbb{R}^2 \to \mathbb{R}^3,\ f(x, y) = (2x + y, x, 2y)$
- $f : \mathbb{R}^3 \to \mathbb{R}^2,\ f(x, y, z) = (2x + y, x + 2z)$