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# Serie 9

1. Show that if $V$ is a one-dimensional vector space over a field $K$ and $T \in \mathrm{Hom}_K(V, V)$, then there exists $\lambda \in K$ such that for all $v \in V : Tv = \lambda v$. Explain then why an isomorphism $V \to K$ depends on a choice of basis, while one from $\mathrm{Hom}_K(V, V)$ to $K$ doesn't.

2. Denote $\mathbb{R}[x]_d$ the set of polynomials over $\mathbb{R}$ of degree lower or equal to $d$. Suppose that $D \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[x]_3, \mathbb{R}[x]_2)$ is the differentiation map $Dp = p'$. Find a basis of $\mathbb{R}[x]_3$ and a basis of $\mathbb{R}[x]_2$ such that the matrix of $D$ with respect to these bases is
$$\begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

3. Let $V, W$ be vector spaces over a field $K$. Suppose that $U \subsetneq V$ is a linear subspace and let $S$ be a non-trivial element of $\mathrm{Hom}_K(U, W)$ (i.e. we assume that $S$ does not map everything to 0). Define $T : V \to W$ by
$$Tv = \begin{cases} Sv, & \text{if } v \in U \\ 0, & \text{if } v \in V \setminus U \end{cases}$$
Is $T$ a linear map?

4. Let $U, V, W$ be vector spaces over a field $K$ and let $T : V \to W$ and $S : W \to U$ be linear maps.
   (a) Prove that
   $$\mathrm{rank}(S \circ T) \leqslant \min(\mathrm{rank}(S), \mathrm{rank}(T)).$$
   (b) Show that $\mathrm{rank}(S \circ T) = \mathrm{rank}(S)$ whenever $T$ is surjective.
   (c) Show that $\mathrm{rank}(S \circ T) = \mathrm{rank}(T)$ whenever $S$ is injective.

5. Let $V$ be a vector space. An Endomorphism $P : V \to V$ satisfying $P^2 := P \circ P = P$ is called idempotent or a projection. Show:
   (a) For ever projection $P$, its image $\Im(P)$ is a linear complement of $\mathrm{Kern}(P)$ in $V$.
   (b) For any subvectorspaces$W_1, W_2 \subset V$, such that $W_1$ is a complement of $W_2$ in $V$, there exists a unique projection $P : V \to V$ with
   $$\mathrm{Kern}(P) = W_1 \quad \text{und} \quad \mathrm{Bild}(P) = W_2.$$

6. Let $f : V \to W$ be a linear map of $K$-vector spaces. Show:

   (a) For every subvectorspace $W' \subset W$ the preimage

   $$f^{-1}(W') := \{v \in V \mid f(v) \in W'\}$$

   is a subvectorspace of $V$.

   (b) We have

   $$\dim f^{-1}(W') = \dim \operatorname{Kern}(f) + \dim (\operatorname{Bild}(f) \cap W').$$

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**Exercises that will not be presented**

7. Let $V, W$ be vector spaces over a field $K$ and let $T : V \to W$ be an isomorphism of vector spaces. Show that:

   (a) $T$ maps linearly independent sets to linearly independent sets;

   (b) $T$ maps spanning sets of $V$ to spanning sets of $W$;

   (c) $T$ maps bases to bases.

8. Let $V, W$ be vector spaces over $\mathbb{Q}$. We say that a map $f : V \to W$ is additive, if

   $$\forall x \in V \, \forall y \in V : f(x + y) = f(x) + f(y).$$

   Show that

   $$\operatorname{Hom}_{\mathbb{Q}}(V, W) = \{f : V \to W \mid f \text{ is additive}\}.$$

**Multiple Choice Questions.** More than one answer can be correct.

**Question 1.** Let $\mathcal{A}$ and $\mathcal{B}$ be bases of $\mathbb{R}^2$ and denote $\{e_1, e_2\}$ the standard basis. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to $\mathcal{A}$ as a basis of the domain and $\mathcal{B}$ as a basis of the codomain. Which of the following statements are true?

- If $\mathcal{A} = \mathcal{B} = \{e_1, e_2\}$, $f$ is a rotation around the origin.
- If $\mathcal{A}$ is the standard basis and $\mathcal{B} = \{e_2, -e_1\}$, $f$ is a symmetry with respect ot the point $(0, 0)$.

- If $\mathcal{A}$ is the standard basis and $f$ is the identity, then $\mathcal{B} = \{-e_2, e_1\}$.
- If $\mathcal{B}$ id the standard basis and $f$ is the symmetry with respect to the $y$-axis, then $A = \{e_2, e_1\}$.

**Question 2.** Which of the following statements are true?

- Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$. The map $V \to \mathbb{R}^n$, $v \mapsto [v]_{\mathcal{B}}$, that sends any vector $v \in V$ to its coordinate vector with respect to a basis $\mathcal{B}$ of $\mathbb{R}^n$ is linear.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $\text{Kern}(f) \neq \{0\}$ and $\text{Bild}(f) \neq \{0\}$. Then there exists a non-trivial vector $v$ that is in the kernel and in the image of $f$.
- If the kernel of a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is trivial, the map is invertible.