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Serie 10

1. Consider the real vector space $M_{2\times 2}(\mathbb{R})$.

   (a) Compute the square of
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   (b) Find a formula for the entries of the n-th power of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{N}$.

2. Let $\mathbb{R}[X]_n$ be the vectorspace of all polynomials of degree $\leqslant n$ with real coefficients.

   (a) Show that
   $$F : \mathbb{R}[X]_n \to \mathbb{R}[X]_n, \quad p \mapsto p'' + p'$$
   is a linear map, where p' denotes the derivative of p.

   (b) Determine the matrix of F with respect to the basis $(1, x, \ldots, x^n)$ of $\mathbb{R}[X]_n$.

3. Let K be a field.

   (a) Consider the matrices
   $$A = \begin{pmatrix} A_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & A_2 \end{pmatrix} \in M_{n\times n}(K)$$
   with $A_1 \in M_{k\times k}(K)$ and $A_2 \in M_{(n-k)\times(n-k)}(K)$ for some $k \geqslant 1$, and
   $$B = \begin{pmatrix} B_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & B_2 \end{pmatrix} \in M_{n\times n}(K)$$
   with $B_1 \in M_{k\times k}(K)$ and $B_2 \in M_{(n-k)\times(n-k)}(K)$. Show that
   $$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & A_2 \cdot B_2 \end{pmatrix}$$

   (b) Let A be defined as above and assume that A_1, respectively A_2, is invertible as an element of $M_{k\times k}(K)$, respectively $M_{(n-k)\times(n-k)}(K)$. Show that A is invertible.

   (c) Consider the space U of upper triangular matrices in $M_{n\times n}(K)$. Show that the product of 2 elements of U is in U.

4. Let V and W be finite-dimensinal vector spaces over a field K.

(a) Suppose that $2 \leqslant \dim V \leqslant \dim W$. Show that

$$\{T \in \operatorname{Hom}(V, W) \mid T \text{ is not injective}\}$$

is not a subspace of $\operatorname{Hom}(V, W)$.

(b) Suppose that $\dim V \geqslant \dim W \geqslant 2$. Show that

$$\{T \in \operatorname{Hom}(V, W) \mid T \text{ is not surjective}\}$$

is not a subspace of $\operatorname{Hom}(V, W)$.

5. Consider the linear maps $\mathbb{R}^4 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$ given by

$$f : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_4 \end{pmatrix} \quad \text{and} \quad g : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 \end{pmatrix}.$$

Let

$$\mathcal{A} := \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right),$$

and let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$ and let

$$\mathcal{C} := \left(\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right).$$

(a) Show that $\mathcal{A}$ is a basis of $\mathbb{R}^4$ and that $\mathcal{C}$ is a basis of $\mathbb{R}^3$.

(b) Determine $g \circ f$ and the matrices

 (i) of f with respect to the bases $\mathcal{A}, \mathcal{B}$.

 (ii) of g with respect to the bases $\mathcal{B}, \mathcal{C}$.

 (iii) of $g \circ f$ with respect to the bases $\mathcal{A}, \mathcal{C}$.

6. Let V be a vector space over a field K. Suppose that T_1 and T_2 are two linear maps from V to K that have the same kernel. Show that there exists a constant $c \in K$ such that $T_1 = cT_2$.

Hint: Use Serie 9 exercise 1.