D-MATH | Lineare Algebra I | HS 2022

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Serie 11

1. Suppose that V, W are finite-dimensional vector spaces over a field K. Let $T \in \operatorname{Hom}(V, W)$. Prove that $\operatorname{rank}(T) = 1$ if and only if there exists a basis $\mathcal{B}$ of V and a basis $\mathcal{C}$ of W such that with respect to these bases, all entries of $[T]^{\mathcal{B}}_{\mathcal{C}}$ equal 1.

2. Suppose V is finite-dimensional and $S, T, U \in \operatorname{Hom}(V, V)$ with $STU = \operatorname{Id}_V$. Show that T is invertible and that $T^{-1} = US$.

3. Let V be a finite-dimensional vector spaces over a field K. Suppose that $T \in \operatorname{Hom}(V, V)$, and that $\mathcal{A} = \{u_1, \ldots, u_n\}$ and $\mathcal{B} = \{v_1, \ldots, v_n\}$ are bases of V. Prove that the following are equivalent:

   (a) T is invertible.

   (b) The columns of $[T]^{\mathcal{A}}_{\mathcal{B}}$ are linearly independent in K^n_{col};

   (c) The columns of $[T]^{\mathcal{A}}_{\mathcal{B}}$ span K^n_{col};

   (d) The rows of $[T]^{\mathcal{A}}_{\mathcal{B}}$ are linearly independent in K^n_{row};

   (e) The rows of $[T]^{\mathcal{A}}_{\mathcal{B}}$ span K^n_{row}.

4. Let V be a finite-dimensional vector space. Prove or disprove:

   (a) Let $V' \subset V$ be a linear subspace. Every automorphism $f : V' \to V'$ can be extended to an automorphism $\bar{f} : V \to V$.

   (b) For every Endomorphism $f : V \to V$ its image $\operatorname{Im}(f)$ is a linear complement $\operatorname{Ker}(f)$ in V.

   (c) There does not exist any linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

   $$\operatorname{rank}(T) = \dim \operatorname{Ker}(T).$$

5. Consider the space $M_{2\times 2}(K)$ of 2-by-2 matrices over a field K.

   (a) Show that if $A \in M_{2\times 2}(K)$ satisfies $A^2 \neq 0$, then $A^k \neq 0$ for all $k \geqslant 3$.

   (b) Find a field K and a matrix $A \in M_{2\times 2}(K) \setminus \{0\}$ such that $\exists n \in \mathbb{N} : A^n = 0$.

6. Determine the ranks of the following rational $n \times n$-matrixes, those being elements of $M_{n\times n}(\mathbb{Q})$, depending on the positive integer n.

   (a) $(kl)_{k,l=1,\ldots,n}$;

   (b) $\big((-1)^{k+l}(k + l - 1)\big)_{k,l=1,\ldots,n}$;

(c) $\left(\frac{(k+l)!}{k!l!}\right)_{k,l=0,\ldots,n-1}.$

Hint: Note that the last matrix is indexed from 0 to $n-1$. You may use induction to find the desired formula.