Serie 12

1. For each of the following matrices, determine whether or not it is invertible, and if it is compute its inverse.

   (a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 6 \end{pmatrix}$

2. Show that a square matrix A is invertible if and only if its transpose A^T is invertible. Moreover, show that in that case we have $(A^T)^{-1} = (A^{-1})^T$.

3. (a) Let V, W be two n-dimensional vector spaces over a field K. Let $S \in \mathrm{Hom}(V, W)$ and $T \in \mathrm{Hom}(W, V)$ such that $T \circ S = \mathrm{Id}_V$. Show that S is invertible and T is the inverse of S.

   (b) Let $A, B \in M_{n\times n}(K)$ such that $A \cdot B = I_n$. Show that A is invertible with inverse B.

4. Consider $n \times n$-matrices A and B over K.

   (a) Show: If A or B is invertible, then AB and BA are similar.

   (b) Does this also hold true without the condition in (a)?

5. Determine with the help of Gaussian elimination, for which values of α the following matrix over $\mathbb{Q}$ is invertible:

$$\begin{pmatrix} 1 & 3 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ 3 & -1 & -2 & -2\alpha \\ -6 & 2 & 1 & \alpha^2 \end{pmatrix}.$$

6. Prove the following:

   Theorem. *(Bruhat-Decomposition)* For every invertible matrix A there exists a permutation matrix P, i.e. a matrix which has exactly one non-zero, which equals 1 in every column and every row, and invertible upper triangular matrices B and B', such that
$$A = BPB'.$$

   Hint: Choose an invertible upper triangular matrix U, such that the sum of the number of leading zeros in all rows of UA is maximal. Then find a permutation matrix Q, such that QUA is an invertible upper triangular matrix.

   Multiplying a matrix A on the left by a permutation matrix P permutes the rows of A. For example, if P has a 1 at index (i, j), the j–th line of A will become the i-th line of PA.

Single Choice. In each exercise, exactly one answer is correct.

1. Which assertion is not always satisfied?
   (a) The base change matrix is the representation matrix of the identity map with respect to the according bases.
   (b) Every finite dimensional vector space is isomorpic to K^n for some $n \geqslant 0$.
   (c) The rank of a linear map $f : K^n \to K^m$ is at least $\min\{n, m\}$.
   (d) The representation matrix of an isomorphism is invertible.

2. Consider $\mathbb{C}$ as real two-dimensional vectorspace with the ordered basis $\mathcal{B} := (1, i)$. The matrix $[\dots]^{\mathcal{B}}_{\mathcal{B}} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the representation matrix with respect to $\mathcal{B}$ of the linear map $\mathbb{C} \to \mathbb{C}$:
   (a) Complex conjugation $z \mapsto \bar{z}$
   (b) $z \mapsto \mathrm{Re}(z)$
   (c) $z \mapsto \mathrm{Im}(z)$
   (d) $z \mapsto iz$

3. The rank of $\begin{pmatrix} 1 & 3 \\ -3 & -9 \end{pmatrix}$ over $\mathbb{Q}$ is

   (a) 0
   (b) 1
   (c) 2
   (d) 3

4. For every $n \times m$-Matrix A and every invertible $n \times n$-matrix B, we have

   (a) $\text{rank}(BA) = \text{rank}(B) \cdot \text{rank}(A)$
   (b) $\text{rank}(BA) = \text{rank}(B) + \text{rank}(A)$
   (c) $\text{rank}(BA) = \text{rank}(A)$
   (d) $\text{rank}(BA) = \text{rank}(B)$

Multiple Choice Fragen.

1. Consider the following ordered bases of $\mathbb{R}[x]_2$:

$$\mathcal{B} = (1, x, x^2), \quad \mathcal{C} = (x^2, (x+1)^2, (x+2)^2)$$

   Determine the base change matrix $Q := [\text{Id}_{\mathbb{R}[x]_2}]^{\mathcal{B}}_{\mathcal{C}}$.

   (a) $Q = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

   (b) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}$

   (c) $Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 4 & 1 \end{pmatrix}$

   (d) $Q = \frac{1}{4} \begin{pmatrix} 2 & -4 & 4 \\ -1 & 4 & -3 \\ 4 & 0 & 0 \end{pmatrix}$

   (e) $Q = \frac{1}{4} \begin{pmatrix} 2 & -3 & 4 \\ -4 & 4 & 0 \\ 2 & -1 & 0 \end{pmatrix}$

   (f) $Q = \frac{1}{4} \begin{pmatrix} 2 & -1 & 4 \\ -4 & 4 & 0 \\ 4 & -3 & 0 \end{pmatrix}$

2. Let $A \in M_{n\times n}(K)$ and $P \in \mathrm{GL}_n(K)$, denote

$$r := \mathrm{rank}(A - I_n) \quad \text{and} \quad s := \mathrm{rank}(PAP^{-1} - I_n).$$

Which of the following statements hold?

(a) $r = s$;

(b) $r \neq s$;

(c) $r > s$;

(d) $r < s$;

(e) if $r < n$, $\exists v \in V \setminus \{0\}$ such that $Av = v$.