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Serie 13

This exercise sheet is to be handed in the week before the Frühjahr Semester 2023.

1. Consider the linear subspace

$$U := \left\langle (2,2,2,2,2)^T, (1,2,2,2,2)^T, (1,1,2,2,2)^T \right\rangle$$

of $V := \mathbb{R}^5$. Determine a subset of the standard basis of $\mathbb{R}^5$, which maps bijectively to a basis of V/U.

2. Let V, W be vector spaces over a field K. Let $U \subseteq V$ a linear subspace, and consider a linear map $f : V \to W$ with $U \subseteq \operatorname{Ker}(f)$. Moreover, consider the linear map induced by the universal property of the quotient vector space

$$\begin{array}{cccc} \bar{f}: & V/U & \to & W \\ & v+U & \to & f(v) \end{array}$$

Show:

(a) $\operatorname{Ker}(\bar{f}) = \operatorname{Ker}(f)/U$.

(b) $\bar{f}$ is injective iff $U = \operatorname{Ker}(f)$.

(c) $\bar{f}$ is surjective iff f is surjective.

(d) If f is surjective, then f induces an isomorphism $V/\operatorname{Ker}(f) \xrightarrow{\sim} W$.

The abbreviation 'iff' is short for 'if and only if' and is very common in mathematical texts.

3. Suppose T is a function from V to W. The graph of T, denoted $\Gamma(T)$, is the subset of $V \oplus W$ defined by

$$\Gamma(T) = \{(v, Tv) \in V \oplus W : v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a linear subspace of $V \oplus W$.

Remark. Formally, a function T from V to W is a subset T of $V \oplus W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then the exercise above could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \oplus W$.

4. Let V be a finite-dimensional vector space over a field K.

   (a) Let $U \subseteq V$ be a subspace and denote W one of its linear complements. Define an isomorphism between V and $U \oplus W$.

   (b) Show that any linear map $\alpha : U \to K$ can be extended to a linear map $\tilde{\alpha}$ on the whole of V. Does $\tilde{\alpha}$ depend on a choice of complement of U?

   (c) Define an isomorphism
$$V^* \cong U^* \oplus W^*.$$

5. Let $(v_1, \ldots, v_n)$ be an ordered basis of a vector space V, and let $(v_1^*, \ldots, v_n^*)$ the corresponding dual basis of the dual vector spaceV^*, and let $((v_1^*)^*, \ldots, (v_n^*)^*)$ the dual basis corresponding to B^* of the bidual vecot space $(V^*)^*$. Show, that the natural isomorphism
$$\tau : V \xrightarrow{\sim} (V^*)^*, \quad v \mapsto \tau(v)$$
maps every v_j to the corresponding $(v_j^*)^*$.

6. Let U, V, W_1 and W_2 be finite-dimensional vector spaces over a field K. Denote by n the dimension of V. Show that:

   (a) $\mathrm{Hom}(V, W_1 \oplus W_2) \cong \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2)$

   (b) $\mathrm{Hom}(V \oplus U, W_1) \cong \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(U, W_1)$

   (c) $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(W^*, V^*)$

   (d) The following map is an isomorphism
$$\begin{array}{ccc} \mathrm{Hom}(V,V) & \to & \mathrm{Hom}(V,V)^* \\ T & \mapsto & [S \mapsto \mathrm{tr}(S \circ T)], \end{array}$$
   where $\mathrm{tr} \in \mathrm{Hom}(V,V)^*$ is the *trace map* defined by
$$\mathrm{tr}(T) = \mathrm{tr}([T]^{\mathcal{B}}_{\mathcal{B}}) = \sum_{i=1}^{n} ([T]^{\mathcal{B}}_{\mathcal{B}})_{ii},$$
   for any basis $\mathcal{B}$, $\mathcal{B}$ of V.

   Remark. You may use this in the above exercise, but as a bonus you could first show that the map $T \in \mathrm{Hom}(V,V) \mapsto \mathrm{tr}(T)$ is independent of the choice of basis of V.

Single Choice. In each exercise, exactly one answer is correct.

1. Let $\dim V = 4$. Then there exists $\varphi \in V^*$ with $\dim \operatorname{Ker} \varphi = 2$.
   (a) Correct
   (b) False
2. Every finite dimensional vector space is the dual of another finite dimensional vecor space.
   (a) Correct
   (b) False
3. The set of all invertible $n \times n$-matrices is...
   (a) not a real linear subspace of $M_n(\mathbb{R})$
   (b) a real linear subspace of $M_n(\mathbb{R})$
4. Let $f : V \to W$ be an arbitrary homomorphism between two K-vector spaces. Which of the following five assertions is not equivalent to the others?
   (a) f is injective.
   (b) The dual map $f^* : W^* \to V^*$ is surjective.
   (c) The zero element of V is the only element mapped to the zero element of W.
   (d) There exists a Homomorphism $g : W \to V$ with $f \circ g = \mathrm{id}_W$.
   (e) For every $v \in V \backslash \{0\}$ there exists $\ell \in W^*$ with $\ell(f(v)) \neq 0$.
   (f) All five assertions are equivalent.

Multiple Choice Fragen.

1. For what value of parameter x is matrix $A = \begin{pmatrix} 1 & x & 1 \\ 3 & 3 & x \\ 0 & 3 & 1 \end{pmatrix}$ not invertible?

    (a) 0

    (b) 1

    (c) 2

    (d) 3

    (e) 4