D–MATH

Exam Linear Algebra I

401-1151-00L

Last Name | *First Name* | *Legi-Nr.*

BÜ SU

XX-123-456

Exam-No.

001

Please do not turn the page yet!

Please take note of the information on the answer-booklet.

Question 1

Determine whether each of the following statements is true or false.
Throughout this exercise, we let V, W, and W' be vector spaces over a field K, and let $U \subseteq V$ be a subspace of V.

1.MC1 [1 Point] Let W_1 and W_2 be linear complements of U in V. Then, W_1 is isomorphic to W_2.

(A) True

(B) False

1.MC2 [1 Point] Let $f : V \to W$ and $g : W \to W'$ be linear maps. Then $g \circ f \equiv 0$ implies $f \equiv 0 \vee g \equiv 0$.

(A) True

(B) False

1.MC3 [1 Point] Consider a subspace $V' \subseteq V$ such that $U \subseteq V'$. Then

$$\left. {}^{V/U} \middle/ {}_{V'/U} \right. \cong \left. {}^{V} \middle/ {}_{V'} \right. .$$

(A) True

(B) False

1.MC4 [1 Point] The matrix

$$\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$$

is invertible.

(A) True

(B) False

1.MC5 [1 Point] The set

$$\{1, x+1, (x+1)^2, (x+1)^3\} \subset K[x]$$

is a basis for $K[x]_3$, the space of polynomials over K of degree at most 3.

(A) True

(B) False

1.MC6 [1 Point] Let $V_1, V_2 \subseteq V$ be subspaces. We have

$$U + (V_1 \cap V_2) = (U + V_1) \cap (U + V_2).$$

(A) True

(B) False

1.MC7 [1 Point] The set

$$\{(t,0,1),(0,t^2,1),(1,0,t)\} \subset \mathbb{R}^3$$

is linearly independent for all $t \in \mathbb{R}$.

(A) True

(B) False

1.MC8 [1 Point] Let $f : V \to W$ and $g : W \to W'$ be linear maps. Assume that f is surjective. We have

$$\operatorname{rank}(g \circ f) = \operatorname{rank}(g).$$

(A) True

(B) False

1.MC9 [1 Point] Let $f : V \to W$ be an injective linear map. Then its dual map $f^* : W^* \to V^*$ is also injective.

(A) True

(B) False

1.MC10 [1 Point] Consider a linearly independent subset $\{v_1, v_2, \cdots, v_m\} \subset V$, for some natural number $m \geq 1$, and let $w \in V$. Then

$$\dim \operatorname{Sp}(v_1 + w, v_2 + w, \cdots, v_m + w) < m.$$

(A) True

(B) False

Question 2

Let V, W be vector spaces over a field K.

2.Q1 [1 Point] Give the definition of $\mathrm{Hom}(V, W)$.

2.Q2 [9 Points] Fix a subspace $U \subseteq V$ and consider the space

$$H := \{f \in \mathrm{Hom}(V, W) \mid f|_U \equiv 0\}.$$

Show that H is isomorphic to $\mathrm{Hom}(V/U, W)$.

Remark. None of the spaces above are assumed to be finite-dimensional.

Question 3

Let K be a field and denote $K[x]_3$ the space of polynomials with coefficients in K of degree at most 3. Fix some $a \in K$ and consider the linear maps

$$\begin{array}{rccc} \mathrm{Comp}: & K[x]_3 & \to & K[x]_3 \\ & p(x) & \mapsto & p(ax) \end{array}$$

and

$$\begin{array}{rccc} D: & K[x]_3 & \to & K[x]_3 \\ & p(x) & \mapsto & p'(x) \end{array}$$

3.Q1 [6 Points] Write the matrices representing Comp and D, respectively, with respect to the standard basis for $K[x]_3$.

3.Q2 [4 Points] Write the matrix representing $D \circ \mathrm{Comp}$ with respect to the standard basis of $K[x]_3$.

Question 4

Let V and W be vector spaces over a field K. Consider a linear map $f : V \to W$.

4.Q1 [**2 Points**] Define the rank of f.

4.Q2 [**8 Points**] Assume that V is finite-dimensional. State and prove the formula relating the dimension of V to the dimension of the kernel of f and the rank of f.

Question 5

Let V be a vector space over a field K.

5.Q1 [**2 Points**] Give the definition of a basis of V.

5.Q2 [**8 Points**] Assume that V is finite-dimensional. Let S be a generating set for V and let $T \subseteq S$ be a finite linearly independent set. Write a short algorithm or explain in a few sentences how we can obtain a basis $\mathcal{B}$ of V that verifies $T \subseteq \mathcal{B} \subseteq S$.

Question 6

Consider the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & \alpha & 3 \\ 3 & \alpha & 2 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}), \quad \alpha \in \mathbb{R}.$$

6.Q1 [**6 Points**] For which values of α is the matrix A invertible?

6.Q2 [**4 Points**] When A is not invertible, give a basis for the kernel of the linear map

$$\begin{array}{rccc} T_A: & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & v & \mapsto & A \cdot v \end{array}$$

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Answer Booklet – Linear Algebra I

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Last Name	*First Name*	*Legi-Nr.*
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001

Exam Duration: 3 hours.

Allowed aids: Dictionary mother tongue – German or English

Please note the following:

- Turn off any **mobile devices and smartwatches** and keep them stored out of reach. **You may not carry any smart devices on you during the exam.**
- Place your Student ID (Legi) visibly on the talbe.
- Only non-erasable pens are allowed. The ink must not be red or green. Except in the True/False exercise, do not use whiteout, instead just cross out the relevant parts.
- Please note the **additional information on the next page**.

Good Luck!

Please do not fill the table!

	1	2	3	4	5	6	Summe
Punkte							
Kontrolle							
Maximal	10	10	10	10	10	10	60

For answering questions:

- Questions should be **answered in this booklet**.
  - Circle your answers in the True/False exercise. You are allowed to use whiteout to make changes.
  - Use the pages designated for a given question.
  - Intermediate steps may be marked! You should simplify your final answer.
  - In case you need additional space you will be provided you with additional paper. Please include a clear indication in case you answer a question somewhere other than on the designated pages of the answer booklet. Remember to mark each additional page you attach with your anonymized code from the front page.

At the end of the exam

- Sort all additional pages by question.
- Wait until all exams have been collected and follow the instructions.

Please do not remove the staple from the booklet.

Question 1

1. (A) True
   (B) False

2. (A) True
   (B) False

3. (A) True
   (B) False

4. (A) True
   (B) False

5. (A) True
   (B) False

6. (A) True
   (B) False

7. (A) True
   (B) False

8. (A) True
   (B) False

9. (A) True
   (B) False

10. (A) True
    (B) False

Question 2

1.

2.

Question 3

1.

2.

Question 4

1.

2.

Question 5

1.

2.

Question 6

1.

2.