Chapter 1

Introduction

1.1 A Nice Introduction

1.1.1 Fibonacci

In this introduction we shall solve a problem with the help of linear algebra, which annoyed me as a teenager.

In middle school we learned about sequences. Consider the following arithmetic sequence: For two numbers a, d define

$$\begin{cases} a_0 = a \\ a_n = a_{n-1} + d, \quad n \geq 1 \end{cases}.$$

Here, one can fairly easily find a formula for the n-th element of the sequence. To be more specific, this means that we can find an expression for the n-th element of the sequence which solely depends on n. In this case, the expression is given by $a_n = a + nd$ for $n \geq 0$. If we would like to, we could easily determine a_{12345} in under half a minute without using a calculator.

We can apply a similar principle to the geometric sequence:

Let a, q be two numbers, then define

$$\begin{cases} a_0 = a \\ a_n = a_{n-1}q, \quad n \geq 1 \end{cases}.$$

In this case a_n is given by $a_n = aq^n$, where n is a variable and ≥ 0.

For me, the interesting part is basically over. One can "prove" some identities, which are mostly tautological, regarding the sums of these sequences (and that might still be slightly interesting). But actually one can pose many far more interesting questions e.g. about arithmetic sequences. For example:

1. Let $a_n = a + nd$ be an arithmetic sequence. Does this sequence contain infinitely many prime numbers? That is, do there exists infinitely many $n \in \mathbb{N}$ such that a_n is prime?

   Assuming a, d are not coprime (i.e. there exist $l \geq 1$ such that l divides a and d), then the sequence contains at most one prime number.

   Theorem 1.1.1 (Dirichlet, 1837). *Let $a, d \in \mathbb{N}, a \geq 1, d > 1$ be coprime. Then the sequence $a_n = a + nd$ contains infinitely many primes.*[1].

   For the proof of the theorem one (more or less) needs a bachelor in mathematics (this could be a topic for a bachelor thesis). If you do not want to wait till then, start here: [2].

2. Because of Dirichlet's Theorem (Theorem 1.1.1) we can ask when does the first prime number in an arithmetic sequence appear. In 1944, Yuri Vladimirovich Linnik, a mathematician who greatly influenced my own research, proved that there exist constants $c, L > 0$ such that the first prime number in an arithmetic sequence $a + nd$, where a, d are coprime (and $1 \leq a < d$) is smaller than cd^L. Take some time to fully understand this statement.

3. Now we can ask whether we can find a "chain" of primes in any arithmetic sequence. What exactly we mean by that you can look up here.

These questions and theorems are indeed interesting, but their solution does not have much to do with linear algebra, so let us take a look at another sequence – the Fibonacci sequence defined as follows:

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_n = a_{n-1} + a_{n-2}, \quad n \geq 2 \end{cases} \tag{1.1}$$

If we compare this sequence with those above, one particular question immediately comes to mind: Can we find an explicit expression for the n-th element of the sequence?

In fact, it is not at all obvious that such an expression even exists, but as a teenager I did hear about it. But when I asked, nobody could tell me what exactly it was or how to find it (back in the old days we did not have Wikipedia ...). Together, we will

[1]For example there are infinitely many primes of the form $1 + 4k$ for $k \in \mathbb{N}$ and infinitely many of the form $3 + 4k$ for $k \in \mathbb{N}$.

now find this expression and in one go get to know nearly all terminology and topics of this lecture course (well, at least of the first semester). The first step is very counter intuitive – we will make matters more complicated. We have no idea how to solve this problem and will look at infinitely many other (related) problems, which we also have no idea how to solve... seems crazy, does it not?

To introduce these new problems, we will develop some mathematical language. This means we will introduce some terminology and definitions. We will regard sequences as objects and will denote them by these lovely calligraphic letters. For example, we will write: Let $\mathcal{F}$ denote the Fibonacci-sequence, defined in (1.1)

Definition 1.1.2. We say that we *know* a sequence *well* if we have found a formula for its n-th element.

For example, we know all arithmetic and geometric sequences well. We can now express our goal as follows: We want to know $\mathcal{F}$ well. As promised, we will make things more complicated:

Definition 1.1.3. Let $a, b \in \mathbb{R}$ be two real numbers. We define the sequence $\mathcal{F}_{a,b}$ by recursion:

$$\begin{cases} a_0 = a \\ a_1 = b \\ a_n = a_{n-1} + a_{n-2}, \text{ for } n \geq 2 \end{cases}$$

Taking $a = 0, b = 1$ we get the standard Fibonacci-sequence $\mathcal{F}$. So our goal in fact is to know $\mathcal{F}_{0,1}$ well. The key step is to define now a completely crazy new goal:

New goal: We want to know $\mathcal{F}_{a,b}$ well for all $a, b \in \mathbb{R}$.

How can it be that we have so much more hope of solving the new goal than the original one? It might seem insane at first, as the new goal actually encompasses infinitely many variants of the old problem. The new problem is a space of problems and we can use this to our advantage! Namely, this space has a certain structure and this gives us some nice properties to work with. Before we start, we just introduce one more definition:

Definition 1.1.4. A sequence $\mathcal{A}$ is called a Fibonacci-sequence if there are $a, b \in \mathbb{R}$ such that $\mathcal{A} = \mathcal{F}_{a,b}$. We denote by **Fib** the space of all Fibonacci-sequences.

Exercise 1.1.5. Show that a sequence $(a_0, a_1, \ldots)$ is a Fibonacci-sequence if and only if $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$.

We have

$$\mathbf{Fib} = \{\mathcal{F}_{a,b} \mid a, b \in \mathbb{R}\}$$

and our goal is to know all of the elements of **Fib** well.

Which structure does Fib have?

Take two sequences $\mathcal{F}_1 = (a_0, a_1, a_2, a_3, \ldots)$, $\mathcal{F}_2 = (b_0, b_1, b_2, b_3, \ldots)$ and add them component-wise

$$\mathcal{F}_1 + \mathcal{F}_2 := (a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots).$$

Claim: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Fibonacci-sequences, then $\mathcal{F}_1 + \mathcal{F}_2$ is also a Fibonacci-sequence. Let us denote the elements of $\mathcal{F}_1 + \mathcal{F}_2$ by $\mathcal{F}_1 + \mathcal{F}_2 = (c_0, c_1, c_2, c_3, \ldots)$. According to Exercise 1.1.5 we only have to show that

$$c_n = c_{n-1} + c_{n-2}$$

for all $n \geq 2$.

So let us prove it: We have

$$c_n = a_n + b_n$$

and as $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Fib}$ it follows that

$$a_n + b_n = (a_{n-1} + a_{n-2}) + (b_{n-1} + b_{n-2}) = (a_{n-1} + b_{n-1}) + (a_{n-2} + b_{n-2}) = c_{n-1} + c_{n-2}.$$

Thus, indeed $c_n = c_{n-1} + c_{n-2}$.

Exercise 1.1.6. With the notation introduced in Definition 1.1.3 convince yourself that the argument above actually shows the following:

$$\mathcal{F}_{a,b} + \mathcal{F}_{c,d} = \mathcal{F}_{a+c,b+d}.$$

To summarize, we can add two elements of $\mathcal{F}$. Hence, space of Fibonacci-sequences **Fib** has an addition. Apart from addition we have another operation in **Fib**: multiplication by a scalar. Here, by a scalar we simply mean a real number. Later in this course we will consider other “types" of numbers, which form a so-called field (maybe you have already heard of the field of real/complex numbers or of a finite field) and a scalar is then simply an element of a field. But for now, a scalar is a synonym for a real number.

So, let $\alpha \in \mathbb{R}$ be a scalar and $\mathcal{A} = (a_0, a_1, a_2, \ldots)$. We define the multiplication of $\mathcal{A}$ by a scalar $\alpha \in \mathbb{R}$ by

$$\alpha\mathcal{A} := (\alpha a_0, \alpha a_1, \alpha a_2, \ldots).$$

Exercise 1.1.7. Arguing similarly as in the addition-case, show that if $\alpha \in \mathbb{R}$ and $\mathcal{A} \in \mathbf{Fib}$, then $\alpha\mathcal{A} \in \mathbf{Fib}$. Furthermore, show that $\alpha\mathcal{F}_{a,b} = \mathcal{F}_{\alpha a, \alpha b}$.

We thus have seen that the space **Fib** has an addition and a multiplication by a scalar. So how does this help us in solving our task of knowing $\mathcal{F}_{0,1}$ well?

Observe that if one knows $\mathcal{F}_1 = (a_0, a_1, \ldots)$ and $\mathcal{F}_2 = (b_0, b_1, \ldots)$ well, then one also knows $\mathcal{F}_1 + \mathcal{F}_2$ well. Indeed, if we have a formula for a_n and b_n, then we also have one for the n-th element of $\mathcal{F}_1 + \mathcal{F}_2$: if $a_n = f(n)$ and $b_n = g(n)$, then $f(n) + g(n)$ is a formula for $\mathcal{F}_1 + \mathcal{F}_2$. Similarly, if one knows a sequence $\mathcal{A} \in \mathbf{Fib}$ well, then one knows $\alpha\mathcal{A}$ also well for all $\alpha \in \mathbb{R}$. More generally, show:

Exercise 1.1.8. If one knows $\mathcal{F}_1, \ldots, \mathcal{F}_k$ well, then one knows

$$\alpha_1\mathcal{F}_1 + \ldots + \alpha_k\mathcal{F}_k \tag{1.2}$$

well for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.

Expressions of the form (1.2) are called linear combinations of sequences.

With the help of this structure of **Fib** we can transfer our knowledge of some elements of **Fib** to other elements of **Fib**.

Well, this all looks nice, but we still do not know any element of **Fib** well! Well, actually this is not entirely true. There is one element we know well.

Exercise 1.1.9. Find it! (Before you continue reading.)

We do know well the Fibonacci-sequence $\mathcal{F}_{0,0}$: if $\mathcal{F}_{0,0} = (a_0, a_1, \ldots)$, then $a_n = 0$ for all $n \geq 0$.

Does this mean that we know many other elements, simply from the structure of **Fib**? Sadly no, we do not, because neither $\mathcal{F}_{0,0} + \mathcal{F}_{0,0}$ nor $\alpha\mathcal{F}_{0,0}$ produce any new sequences... We simply get $\mathcal{F}_{0,0}$ over and over. Hence, if we know only $\mathcal{F}_{0,0}$ well, we cannot achieve anything with addition and multiplication by a scalar. How many sequences do we have to know well in order to know well all elements of **Fib**?

Assume that we know a Fibonacci-sequence $\mathcal{F}_{a,b}$ well, where a and b are not zero. Then via multiplication by a scalar we know $\mathcal{F}_{\alpha a, \alpha b}$ well for all $\alpha \in \mathbb{R}$.

Exercise 1.1.10. Show that the set $\{\mathcal{F}_{\alpha a,\alpha b} \mid \alpha \in \mathbb{R}\} \subseteq \mathbf{Fib}$ under the assumption $(a, b) \neq (0, 0)$ has infinitely many elements but still does not equal **Fib**.

This exercise shows that we need to know at least two Fibonacci-sequences well to know all elements of **Fib** well. Do there exist two sequences in **Fib**, which we know well and with the help of which we then know all elements of **Fib** well? Can we find two sequences $\mathcal{A}, \mathcal{B} \in \mathbf{Fib}$, such that every sequence is a linear combination of $\mathcal{A}$ and $\mathcal{B}$?

If we know $\mathcal{F}_{0,1}$ and $\mathcal{F}_{1,0}$ well for example, then we know all elements of **Fib** well: we can write a general element $\mathcal{F}_{a,b}$ of **Fib** as follows:

$$\mathcal{F}_{a,b} = a\mathcal{F}_{1,0} + b\mathcal{F}_{0,1}$$

Therefore, we can find a formula for $\mathcal{F}_{a,b}$ if we know the formulas for $\mathcal{F}_{0,1}$ and $\mathcal{F}_{1,0}$.

Again, this is very cool, but we still have not solved our original problem! We still do not know $\mathcal{F}_{0,1}$ well! In the following section we will get to know other sequences well and with these we will be able to know well all other elements of **Fib** too.

Remark 1.1.11. The fact that

(1) two sequences suffice,

(2) one sequence does not suffice (Exercise 1.1.10),

(3) in every set of three sequences there exists a sequence we can leave out without changing the set of “achievable sequences",

are connected to the fact that the space **Fib** has dimension 2. This does not come as a surprise. Dimension more or less measures the number of “degrees of freedom" of a space. Convince yourself that **Fib** has two degrees of freedom in the reals. In other words, **Fib** is a plane, in which every point represents a sequence.

1.1.2 Prior Knowledge and Symmetry

How does one come up with an idea for the solution of a problem? Usually, by using prior knowledge, which is relevant to the solution or by recognizing certain patterns or symmetries of the problem.

Perhaps you are thinking that the problem we are trying to solve cannot have any symmetries, as it is not a geometric problem...

Prerequisites

Before we explain what symmetry we are talking about and why it is the key to solving the problem, we shall use our prior knowledge to find Fibonacci-sequences we know well.

Exercise 1.1.12. Show that **Fib** does not contain arithmetic sequences apart from $\mathcal{F}_{0,0}$.

The exercise shows that we cannot advance with arithmetic sequences. How about geometric ones? Can a sequence of the form $(a, aq, aq^2, aq^3, \ldots)$ be a Fibonacci-sequence?

For simplicity, let us first consider $a = 1$, that is consider

$$\mathcal{G}_q = (1, q, q^2, \ldots)$$

where $q \neq 0$. The sequence $\mathcal{G}_q$ is a Fibonacci-sequence, if and only if

$$q^n = q^{n-1} + q^{n-2} \tag{1.3}$$

for all $n \geq 2$. As $q \neq 0$, we can divide (1.3) by q^{n-2} and hence (1.3) is equivalent to

$$q^2 = q + 1. \tag{1.4}$$

This we can solve with other prior knowledge, namely the formula for solving quadratic equations[2] The equation (1.4) is then true if and only if

$$q = \frac{1 \pm \sqrt{5}}{2}.$$

Define:

$$\begin{aligned} \phi &:= \frac{1+\sqrt{5}}{2} \approx 1.618033\ldots \quad \text{(golden section)} \\ \psi &:= \frac{1-\sqrt{5}}{2} \approx -0.618033\ldots \quad \text{(conjugated golden section)}. \end{aligned}$$

Let us summarize what we have established so far. The sequences

$$\mathcal{G}_\varphi = (1, \varphi, \varphi^2, \varphi^3, \ldots) \quad \text{and} \quad \mathcal{G}_\psi = (1, \psi, \psi^2, \psi^3, \ldots)$$

are both elements of **Fib**, which we know well! This is great, but can we express the original sequence $\mathcal{F}_{0,1}$ (via linear combinations of $\mathcal{G}_\varphi$ and $\mathcal{G}_\psi$)? Yes, indeed we can!

Exercise 1.1.13. Verify that

$$\frac{1}{\varphi - \psi}\mathcal{G}_\varphi + \frac{1}{\psi - \varphi}\mathcal{G}_\psi = \mathcal{F}_{0,1}.$$

Remark 1.1.14. To solve Exercise 1.1.13, you probably have to solve a system of linear equations. These are central in linear algebra. One of the first topics we shall discuss is going to be an algorithm to solve systems of linear equations – Gaussian elimination.

Exercise 1.1.13 gives us a formula for the n-th element of $\mathcal{F}_{0,1} = (a_0, a_1, a_2, \ldots)$:

$$a_n = \frac{1}{\varphi - \psi}\varphi^n + \frac{1}{\psi - \varphi}\psi^n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

We have achieved our goal!

Exercise 1.1.15. Let $\mathcal{F}_{0,1} = (F_0, F_1, F_2, \ldots)$. Under the assumption, that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists, compute this limit. This gives us another motivation to study geometric sequences with q as their limit.

[2] Finally a good reason to learn how to solve quadratic equations!

Symmetry

One could argue that what we have done above was based on pure luck. How could we have known that geometric sequences will have proven to be useful? This is a fair point. As hinted at above one could discover these geometric sequences by looking at the symmetry of the space **Fib**. But what exactly does one mean by symmetry in this case? If X is some geometric space, then a symmetry of X is a map[3]

$$T : X \to X,$$

which preserves (or respects) all/some of the geometric properties of X, e.g. distances, angles etc. If $T\colon X \to X$ is a symmetry, the set of fixed points

$$\mathrm{Fix}(T) = \{x \in X : T(x) = x\}$$

usually is an interesting set to study. In other words, a fixed point of T is an element which gets mapped by T to itself. To be able to transfer the terminology of a fixed point to other (more general) spaces, we define the following. Let X be a space with a certain structure. A symmetry of X is a map $: X \to X$, which preserves/respects the structure of X.

If $X = \mathbf{Fib}$ we get the following:

Definition 1.1.16. A map $T\colon \mathbf{Fib} \to \mathbf{Fib}$ is called a symmetry of **Fib**, if for $\mathcal{A}, \mathcal{B} \in \mathbf{Fib}$ and $\alpha \in \mathbb{R}$ we have

$$T(\mathcal{A} + \mathcal{B}) = T(\mathcal{A}) + T(\mathcal{B}) \quad \text{and} \quad T(\alpha\mathcal{A}) = \alpha T(\mathcal{A}). \tag{1.5}$$

The requirements in (1.5) are what we mean by T “respects" the structure of **Fib**.

Let us first think about which maps $T\colon X \to X$ we already know. Here are three rather unexciting examples:

1. The identity map

$$\begin{aligned} \mathrm{Id} : \mathbf{Fib} &\to \mathbf{Fib} \\ \mathcal{A} &\mapsto \mathcal{A}, \end{aligned}$$

which “does not do anything”.

[3] A *map* is another name for a function. A function (or a map) assigns to every element of X a specific element of X. This and other related basic notions will defined carefully and discussed here as well as in the Analysis I course.

2. The “multiplication-by-a-scalar" map: For $\alpha \in \mathbb{R}$ we define

$$\begin{aligned} \mathrm{M}_\alpha : \mathbf{Fib} &\to \mathbf{Fib} \\ \mathcal{A} &\mapsto \alpha\mathcal{A}. \end{aligned}$$

3. The vector addition/sequence addition map: For $\mathcal{B} \in \mathbf{Fib}$ we define

$$\begin{aligned} \mathrm{A}_\mathcal{B} : \mathbf{Fib} &\to \mathbf{Fib} \\ \mathcal{A} &\mapsto \mathcal{A} + \mathcal{B}. \end{aligned}$$

Exercise 1.1.17. Show that Id and M_α for all $\alpha \in \mathbb{R}$ fulfills the requirements in (1.5). Additionally, show that $\mathrm{A}_\mathcal{B}$ for $\mathcal{B} \neq \mathcal{F}_{0,0}$ does **not** fulfill the requirements in (1.5).

As mentioned earlier, these symmetries are not particularly interesting. Perhaps because they are not connected to the fact that **Fib** is a space of sequences. In fact, these map do exist for every space with an addition and multiplication by a scalar.

Exercise 1.1.18. Before you continue reading, try to find a different interesting map $T\colon \mathbf{Fib} \to \mathbf{Fib}$, which takes into account that **Fib** is a space of sequences? (Hint: if one knows $\mathcal{F}_{1,0}$ well, then one also knows $\mathcal{F}_{0,1}$ well. Why?)

In spaces of sequences we also have a translation map

$$\begin{aligned} S : \mathbf{Fib} &\to \mathbf{Fib} \\ (a_0, a_1, a_2, \ldots) &\mapsto (a_1, a_2, \ldots). \end{aligned}$$

Exercise 1.1.19. (1) Verify that $S(\mathcal{A}) \in \mathbf{Fib}$ for $\mathcal{A} \in \mathbf{Fib}$.

(2) Verify that S satisfies the requirements in (1.5).

This translation map turns out to be an interesting symmetry, and it is related to spaces of sequences. Therefore, we can ask ourselves which fixed points does S have.

Exercise 1.1.20. Show that $\mathcal{F}_{0,0}$ is the only fixed point of S.

OK, so the fixed points of S are not so interesting. It turns out that for maps that fulfill (1.5), the set of fixed point mostly is boring. The requirements imposed on $S(\mathcal{A}) = \mathcal{A}$ are simply too limiting to produce interesting sequences. Therefore, let us consider a sightly weaker requirement (which is also connected to multiplication by a scalar).

Definition 1.1.21. Let $T\colon \mathbf{Fib} \to \mathbf{Fib}$ be a symmetry. A sequence $\mathcal{A} \neq \mathcal{F}_{0,0}$ is called an *eigensequence* of T, if there exists an $\alpha \in \mathbb{R}$, such that

$$T(\mathcal{A}) = \alpha A.$$

The scalar α is called the *eigenvalue* of $\mathcal{A}$.

Thus, we want to find all eigensequences[4] of the symmetry S.

To this end, assume that $\mathcal{A} = (a_0, a_1, a_2, \ldots) \neq F_{0,0}$ is an eigensequence with eigenvalue α. Then $S(\mathcal{A}) = \alpha A$ or in other words

$$(a_1, a_2, a_3, \ldots) = \alpha(a_0, a_1, a_2, \ldots) = (\alpha a_0, \alpha a_1, \alpha a_2, \ldots).$$

Therefore, $a_n = \alpha a_{n-1}$ for all $n \geq 1$. Hence, $\mathcal{A}$ has the form

$$\mathcal{A} = (a_0, a_0\alpha, a_0\alpha^2, a_0\alpha^3, \ldots). \tag{1.6}$$

This means that $\mathcal{A}$ is a geometric sequence. So which α (and which a_0) come into question? We have already seen this in Section 1.1.2, but let us repeat this briefly. Because of (1.6) we have

$$a_n = \alpha^n a_0$$

for all $n \geq 0$. As $\mathcal{A} \in \mathbf{Fib}$, we have $a_2 = a_1 + a_0$ and putting these two facts together we arrive at

$$\alpha^2 a_0 = a_2 = a_1 + a_0 = \alpha a_0 + a_0 = (\alpha + 1)a_0. \tag{1.7}$$

From $\mathcal{A} \neq \mathcal{F}_{0,0}$ it follows that $a_0 \neq 0$. (Why? Convince yourself.) Hence, (1.7) is equivalent to $\alpha^2 = \alpha + 1$. Does this seem familiar? This is precisely the equation (1.4). This means: if $\mathcal{A}$ is an eigensequence with eigenvalue α, then $\alpha^2 = \alpha + 1$, hence either $\alpha = \psi$ or $\alpha = \varphi$. That is, $\mathcal{A}$ is a geometric sequence with $\alpha = \varphi$ or $\alpha = \psi$. For simplicity we choose $a_0 = 1$. Then we get the two sequences we guessed earlier

$$\mathcal{G}_\varphi = (1, \varphi, \varphi^2, \varphi^3, \ldots) \quad \text{and} \quad \mathcal{G}_\psi = (1, \psi, \psi^2, \psi^3, \ldots).$$

Summary : The n'th Fibonacci-number F_n is given by

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

More generally, try to solve the following:

Exercise 1.1.22. Find a formula for the n-th element of $\mathcal{F}_{a,b}$ depending on a, b and n.

There are many other interesting things in this context, but it's about time to begin with our course. Here is a small exercise for “dessert":

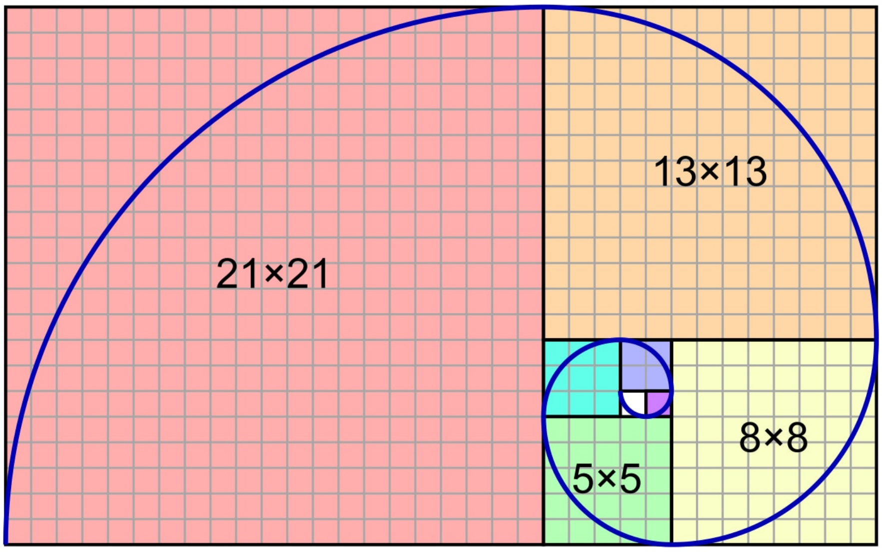
[4] For the advanced reader: eigensequences are actually fixed points with respect to S on the projective space $\mathbb{P}(\mathbf{Fib})$.

Exercise 1.1.23. Show that

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}.$$

What exactly is meant by this notation and how this corresponds to the movement of planets, you can ask me or look up here [1, §1.1].

Remark 1.1.24. The idea for this introduction came to mind, when I was a teaching assistant at the Hebrew University and was asked why linear algebra was fun. I shared this question with the world (of stackexchange) and many of the answers given there are very interesting and worth thinking about!



Figur 1.1: Your new t-shirt?

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