Lecture #1A.

The Fibonacci sequence

(Leonardo Fib.
Mid. age math from
Pisa. (1170 – 1250).
Growth of rabbits)

$a_0, a_1, a_2, a_3, \dots, a_n, \dots$

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2 \end{cases}$$

$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2$
$a_4 = 3, a_5 = 5, a_6 = 8$
$a_7 = 13, a_8 = 21, \dots$

Formula for a_n ?

Let's look at other sequences we know well:

Arithmetic seq.

$$\begin{cases} a_0 = a \\ a_n = a_{n-1} + d, \quad \text{for } n \geq 1 \end{cases}$$

Formula for a_n: $a_n = a + n \cdot d$.

Geometric seq.

$$\begin{cases} a_0 = a \\ a_n = a_{n-1} \cdot q, \quad n \geq 1 \end{cases}$$

Formula for a_n: $a_n = a \cdot q^n$.

But Fibonacci is different...!

Let's generalize the problem. (why?)

Let a_0, a_1 be real numbers. $(a_0, a_1 \in \mathbb{R})$

$$\mathcal{F}_{a_0,a_1}: \left\{ \begin{array}{l} \left.\begin{array}{l} a_0 \\ a_1 \end{array}\right\rangle \text{given} \\ \\ a_n = a_{n-1} + a_{n-2}, \quad n \geq 2 \end{array} \right\} \quad \begin{array}{l} \mathcal{F}_{0,1} \text{ is our} \\ \text{original} \\ \text{Fib. seq.} \end{array}$$

Let's call a seq. $\mathcal{A}$ a Fibonacci seq. if $\mathcal{A} = \mathcal{F}_{a_0,a_1}$ for some a_0, a_1. Denote the collection (set, space) of all Fib. seq. by Fib.

Fib has some algebraic structure

Claim. If $\mathcal{F}_1 = (a_0, a_1, a_2, a_3, \ldots)$ and $\mathcal{F}_2 = (b_0, b_1, b_2, b_3, \ldots)$ are Fib. seq. $\Longrightarrow \mathcal{F}_1 + \mathcal{F}_2 := (a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$ is also a Fibonacci sequence.

Proof. $\mathcal{F}_1 + \mathcal{F}_2 = (c_0, c_1, c_2, c_3, \ldots)$

$$c_n \stackrel{?}{=} c_{n-1} + c_{n-2}. \quad \text{YES:} \left\{ \begin{array}{l} c_n = a_n + b_n \\ c_{n-1} = a_{n-1} + b_{n-1} \\ c_{n-2} = a_{n-2} + b_{n-2} \end{array} \right.$$

$$\begin{aligned} c_{n-1} + c_{n-2} &= a_{n-1} + b_{n-1} + a_{n-2} + b_{n-2} \\ &= (a_{n-1} + a_{n-2}) + (b_{n-1} + b_{n-2}) = a_n + b_n = c_n \end{aligned}$$

This shows that $\mathcal{F}_{a_0,a_1} + \mathcal{F}_{b_0,b_1} = \mathcal{F}_{a_0+b_0,\, a_1+b_1}$.

So Fib is closed under addition.

Let $\mathcal{A} = (a_0, a_1, a_2, a_3, \dots, a_n, \dots)$ be a Fib. seq. $\alpha \in \mathbb{R}$.

Define $\alpha\mathcal{A} = (\alpha a_0, \alpha a_1, \alpha a_2, \alpha a_3, \dots, \alpha a_n, \dots)$

Claim. $\alpha\mathcal{A}$ is also a Fib. seq.

Proof. We need to check that $\alpha a_n = \alpha a_{n-1} + \alpha a_{n-2}$.
But this is obviously true, b.c. $a_n = a_{n-1} + a_{n-2}$.

In fact, we see that $\alpha \mathcal{F}_{a_0,a_1} = \mathcal{F}_{\alpha a_0, \alpha a_1}$.

Summary. Linear combinations of Fib. seq.

$$\underbrace{\alpha \cdot \mathcal{F}_{a_0,a_1} + \beta \mathcal{F}_{b_0,b_1}}_{\mathcal{F}_{\alpha a_0+\beta b_0,\, \alpha a_1+\beta b_1}}, \quad \alpha, \beta \in \mathbb{R}, \text{ are also Fib.}$$

In order to find a formula for all Fib. sequences $\mathcal{F}_{a_0,a_1}$,

it is enough to find formulas for $\mathcal{F}_{1,0}$ & $\mathcal{F}_{0,1}$

b.c. $\mathcal{F}_{a_0,a_1} = a_0 \cdot \mathcal{F}_{1,0} + a_1 \cdot \mathcal{F}_{0,1}$.

We will see that Fib. is a 2-dim. $\underbrace{\text{vector space}}$.

What is a vector space and what is "dim." will be precisely defined in the course.

every element can be expressed by a lin. combin. of some $\underline{\underline{2}}$ elements. But 1 element is not enough

closed under addit. + multip. of a scalar

Perhaps one of the Fibonacci seq. $\mathcal{F}_{a_0,a_1}$ has the form we already know?

But what types of seq. we know?

- Arithmetic seq.: $a_n = a + nd$
- Geometric seq.: $a_n = a \cdot q^n$.

* Arithmetic seq.: no Fibonacci seq. $\mathcal{F}_{a_0,a_1}$, except of $\mathcal{F}_{0,0}$ can be arithmetic.

<u>Reason.</u> For arithm. seq. we have $a_n - a_{n-1} = d = \text{const}$. But for Fib. we have

$a_n - a_{n-1} = a_{n-2}$ and this can't happen for an arithm. seq. (exc. complete the details).

So: arithm. seq. are no good for this purpose.

* Perhaps for some a_0, a_1, $\mathcal{F}_{a_0,a_1}$ coincides with a geometric seq. ?

Let's try the seq. $\mathcal{G} = (1, q, q^2, q^3, \dots, q^n, \dots)$

Can we find some $q \neq 0$, <u>s.t.</u> $\mathcal{G}$ is Fibonacci ?

We need that $q \neq 0$, $q^n = q^{n-1} + q^{n-2}$, $n \geq 2$.

$\iff$ $q^2 = q + 1$. $\iff$ $q = \frac{1 \pm \sqrt{5}}{2}$. Two solutions!

$\varphi := \frac{1+\sqrt{5}}{2}$ ($\approx 1.618033\dots$ the golden ratio)

$\psi := \frac{1-\sqrt{5}}{2}$ ($\approx -0.618033\dots$)

Then $(1, \varphi, \varphi^2, \varphi^3, \dots, \varphi^n, \dots) = \mathcal{F}_{1,\varphi}$

$(1, \psi, \psi^2, \psi^3, \dots, \psi^n, \dots) = \mathcal{F}_{1,\psi}$.

So far we haven't solved our original problem which is to find a formula for $\mathcal{F}_{0,1}$.

But we did find formulas for two Fib. seq.s:

$\mathcal{F}_{1,\varphi}$ & $\mathcal{F}_{1,\psi}$.

<u>But recall that</u>: $\alpha \cdot \mathcal{F}_{a_0,a_1} + \beta \cdot \mathcal{F}_{b_0,b_1} = \mathcal{F}_{\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1}$

so let's try to use this for $a_0 = 1, a_1 = \varphi, b_0 = 1, b_1 = \psi$.

We need α & β s.t.: $\begin{cases} \alpha \cdot 1 + \beta \cdot 1 = 0 \\ \alpha \cdot \varphi + \beta \cdot \psi = 1 \end{cases}$

$\iff \beta = -\alpha, \quad \alpha \cdot \varphi - \alpha \cdot \psi = 1 \Rightarrow \alpha = \frac{1}{\varphi - \psi} = \frac{1}{\sqrt{5}}.$

$\beta = \frac{-1}{\sqrt{5}}.$

So $\mathcal{F}_{0,1} = \frac{1}{\sqrt{5}} \mathcal{F}_{1,\varphi} - \frac{1}{\sqrt{5}} \mathcal{F}_{1,\psi}$

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

But how did we come to the idea to find a geometric seq. among the Fibonacci's ?

Let's look at maps/functions/transfor./operators $T: \mathrm{Fib} \longrightarrow \mathrm{Fib}$

T gets as input a (any) Fib. seq. $\mathcal{F}$ and gives as an output another Fib. seq. $T(\mathcal{F})$.

We saw that Fib. has some alg. struct.: closed under <u>addition</u> & <u>multiplic. by scalars.</u> So let's look at maps T that respect these structures/operations:

$$\begin{cases} T(\mathcal{A}+\mathcal{B}) = T(\mathcal{A}) + T(\mathcal{B}) & \text{for all } \mathcal{A},\mathcal{B} \in \mathrm{Fib} \\ T(\alpha\mathcal{A}) = \alpha T(\mathcal{A}) & \text{for all } \alpha \in \mathbb{R}, \mathcal{A} \in \mathrm{Fib}. \end{cases}$$

(linear map)

There are many examples of such T's : $* \; T = id, \; T(\mathcal{A}) := \mathcal{A}$

$* \; T(\mathcal{A}) := \gamma \cdot \mathcal{A}, \quad \gamma \in \mathbb{R}$

Here is another, more interesting one

$S: \mathrm{Fib} \to \mathrm{Fib}, \quad S(a_0, a_1, a_2, \dots, a_n, \dots) := (a_1, a_2, \dots, a_n, \dots)$

the shift map.

* Check that S is linear (exc.)

Mathematicians like to look for special/extreme properties/values of maps $T: X \longrightarrow X$, for example fixed points : $x_0 \in X$ s.t. $T(x_0) = x_0$.

Another example is: eigenvectors

An element $y_0 \in X$ s.t. $T(y_0)$ is proportional to y_0, i.e. $T(y_0) = \lambda \cdot y_0$ for some $\lambda \in \mathbb{R}$.

Let's check this for $X = \text{Fib}$, $T = S = $ shift map.

<u>Fixed pnts:</u> $S(a_0, a_1, \dots, a_n, \dots) := (a_1, a_2, \dots, a_n, \dots)$

$(a_0, a_1, \dots)$ is a fixed pnt.

means that $a_0 = a_1$, $a_1 = a_2$, $a_2 = a_3$, ...

$\Rightarrow$ $a_n = a_0$ for all n. This can be Fibonacci if and only if the seq. is $(0, 0, 0, \dots, 0, \dots)$

So: NOT interesting.

Perhaps an eigenvector? $S(\mathcal{A}) = \lambda \cdot \mathcal{A}$

$$\underbrace{S(a_0, a_1, a_2, \dots, a_n, \dots)}_{\overset{\shortparallel}{(a_1, a_2, \dots, a_{n+1}, \dots)}} = \lambda \cdot (a_0, a_1, \dots, a_n, \dots)$$

We get $a_n = a_{n-1} \cdot \lambda$ for all $n \geq 1$.

$\Rightarrow \mathcal{A} = (a_0, a_0 \cdot \lambda, a_0 \cdot \lambda^2, a_0 \cdot \lambda^3, \dots, a_0 \cdot \lambda^n, \dots)$

a geometric seq. !

what would a_0 and λ be?

We need $\underbrace{a_0 \cdot \lambda^2}_{a_2} = \underbrace{a_0 \cdot \lambda}_{a_1} + a_0 \Rightarrow \lambda^2 = \lambda + 1$

So $\lambda = \frac{1 \pm \sqrt{5}}{2}$. And a_0 can be chosen as we like.
(e.g. $a_0 = 1$)

Upshot: the two geometric sequences

$\mathcal{A} = (1, \varphi, \varphi^2, \dots, \varphi^n, \dots)$

$\mathcal{B} = (1, \psi, \psi^2, \dots, \psi^n, \dots)$

are Fibonacci & are eigenvectors of the shift operator.

Interesting exercises.

* Find an explicit formula for F_{a_0,a_1} that depends only on a_0, a_1 and n.

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* Consider $F_{0,1} = (a_0, a_1, a_2, \dots, a_n, \dots)$
  calculate $\lim_{n \to \infty} \frac{a_n}{a_{n-1}}$.

* Consider $\varphi = \frac{1+\sqrt{5}}{2}$. Show that

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

what is actually the meaning of this?