# Logic & Math.

Mathematical/logical statement: a statement that can be either True or False (can't be both true & false).

For example:

A : "For every real number $x$, we have $x^2 \geq 0$".

B : " Every prime number $p \geq 3$ must be odd ".

C : " Every odd number $p \geq 3$ must be prime".

Obviously: A is True, B is True, C is False.

Negation of a statement A: This is the statement saying that A does NOT hold.

We write $\neg A$ for this statement (sometimes $\overline{A}$).
(latex: \neg A)

Truth-table:

| $A$ | $\neg A$ |
|---|---|
| T | F |
| F | T |

# And & Or operations.

Let A and B be statements.

(latex: A \wedge B)

$A \wedge B$ is the statement "A holds <u>and</u> B holds".

(Sometimes $A$ & $B$) In other words "Both A and B hold".

## Truth-table:

| $A$ | $B$ | $A \wedge B$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Exp. For real numbers $x$,

$$(\text{"}x^2=1\text{"}) \wedge (\text{"}x \geq 0\text{"})$$

means the same as "$x=1$".

$A \vee B$ means "A <u>or</u> B hold" i.e. "At least one of A, B hold"

every day life (either A or B)

math (either A or B or both)

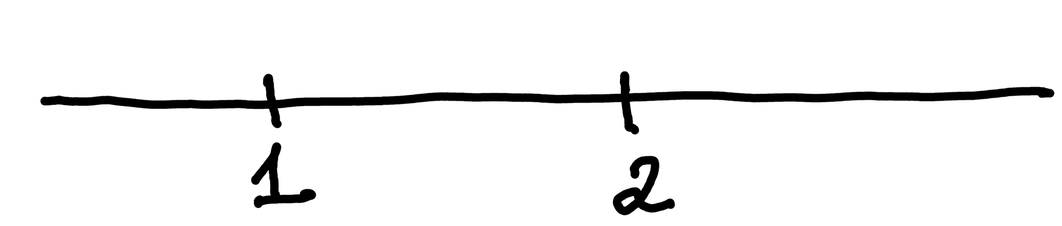
could be only A
could be only B
could be both A and B.

## Truth-table

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Example: $(\text{"}x>1\text{"}) \vee (\text{"}x<2\text{"})$ is always true.

$(\text{"}x<1\text{"}) \vee (\text{"}x>2\text{"})$ is the same as $\neg(\text{"}1 \leq x \leq 2\text{"})$



## Logical implications.

Let $A, B$ be statements. We can form a new statement : "$A \Rightarrow B$". It states that : whenever $A$ holds, then $B$ holds too. (If $A$ is True, then $B$ is True.)

By definition:

| $A$ | $B$ | $A \Rightarrow B$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

So: ("$A \Rightarrow B$") is the same as $(\neg A \vee B)$

Example: ("$0=1$") $\Longrightarrow$ Eearth is flat

is a True statement.

---

## Logical equivalence. $A \Longleftrightarrow B$

$A \Longleftrightarrow B$ is the statement $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

To say that "$A \Longleftrightarrow B$" is True is the same as to say that $A$ holds $\underbrace{\text{if and only if}}_{\text{iff}}$ $B$ holds, or that $A$ and $B$ are equivalent.

Exp. $("x^2>0") \iff ("x \neq 0")$
$("0=1") \iff ("\text{Eearth is flat}")$ } both statem. are true.

⊛ Remark: causality is irrelevant !

---

Let A,B be statements. Then the following holds.

$$(A \Rightarrow B) \iff (\neg B \Rightarrow \neg A)$$

Exc. show that: $\neg(\neg A) = A$

$$\neg(A \wedge B) = \neg A \vee \neg B, \quad \neg(A \vee B) = \neg A \wedge \neg B$$

spell out these two in words.

---

## Predicate logic.

Predicate: a statement involving some variables from a set. $P(x)$, $P(x,y)$, ...

For example $P(n) = "n = n^2"$. It can be true/false depending on the value of the variable.

Here is a False statement: $\underbrace{\text{"for all } n \in \mathbb{Z} : n = n^2\text{"}}_{\forall n \in \mathbb{Z} : n = n^2}$.

$\forall$ = for all/all

Here is a True statement: $\underbrace{\text{"there exist } n \in \mathbb{Z} : n = n^2\text{"}}_{\exists\, n \in \mathbb{Z} \,:\, n = n^2}$.

$\exists$ = there exist(s)/exist(s)

Another true statement: $\forall\, n \in \mathbb{Z} : \left(n = n^2 \Rightarrow (n=0) \vee (n=1)\right)$.

$\forall$ & $\exists$ are called quatifiers.

We can combine them:

$$\forall\, y \in \mathbb{R} : \left(y \geq 0 \Rightarrow \left(\exists\, x \in \mathbb{R} : x^2 = y\right)\right) \qquad \text{(TRUE)}$$

In practice we write it as follows:

$$\forall\, 0 \leq y \in \mathbb{R},\ \exists\, x \in \mathbb{R} \ \underline{\underline{\text{s.t.}}}\ x^2 = y. \qquad (*)$$

Be careful about the order:

$\forall x \in X, \exists\, y \in Y : A(x,y)$ is NOT the same

as $\exists y \in Y, \forall x \in X : A(x,y)$. TRY it on $(*)$.

Another example: $X$ = set of students at ETH
$Y$ = set of courses.

for $x \in X$, $y \in Y$, $A(x,y)$ = student $x$ takes course $y$.

"$\forall x \in X, \exists\, y \in Y : A(x,y)$" is true

But $\underbrace{\exists y \in Y, \forall x \in X}_{\text{there exists a course s.t. all the students of ETH take this course.}} : A(x,y)$ is not true.

<u>Shorthand notation:</u> "$\exists x \in X, \exists y \in X$" $\rightsquigarrow$ "$\exists x, y \in X$"

"$\forall x \in X, \forall y \in X$" $\rightsquigarrow$ "$\forall x, y \in X$".

<u>Negation of quantifiers.</u>

$$\neg (\forall x \in X : A(x)) \iff \exists x \in X : \neg A(x)$$

$$\neg (\exists x \in X : A(x)) \iff \forall x \in X : \neg A(x).$$

---

<u>There exists a unique ...</u>

"$\exists ! x \in X : A(x)$" means:

$(\exists x \in X : A(x)) \wedge (\text{among } y \in X \text{ for which } A(y),\ y = x \text{ is unique})$

Another way to write this:

$$(\exists x \in X : A(x)) \wedge (\forall x, y \in X : (A(x) \wedge A(y) \Rightarrow x = y))$$

In other words: there is an $x \in X$ for which $A(x)$ holds, and this $x$ is the only one.

Example: "$\exists! \, x \in \mathbb{R} : x^2 = 4$" is not true

But "$\exists! \, x \in \mathbb{R} : x \geq 0 \land x^2 = 4$" is true.