Lecture # 4

Systems of linear equations.

Fields (Körper).

* Axioms of a field.

* **Examples:** $\mathbb{Q} = \{ a \mid a = \frac{m}{n},\ m, n \in \mathbb{Z},\ n \neq 0 \}$

$\mathbb{R}$ - real numbers, $\mathbb{C}$ - complex numbers
$= \{ z \mid z = a + i \cdot b,\ a, b \in \mathbb{R} \},\ i \cdot i = -1.$

$\mathbb{F}_2 = \{0, 1\}$ operations are done modulo 2

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

$\mathbb{F}_3 = \{0, 1, 2\}$ operations are done mod 3.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$$\begin{cases} x+3y+4z=-1 \\ 2x-y+z=5 \end{cases} \xrightarrow{-2\cdot E_1+E_2\to E_2} \begin{cases} x+3y+4z=-1 \\ -7y-7z=7 \end{cases}$$

$$\xrightarrow{-\frac{1}{7}\cdot E_2\to E_2} \begin{cases} x+3y+4z=-1 \\ y+z=-1 \end{cases} \xrightarrow{-3\cdot E_2+E_1\to E_1} \begin{cases} x \qquad +z=2 \\ y+z=-1 \end{cases}$$

set of solutions: $z=c\in\mathbb{R}$ an arbit. real number

$y=-1-c, \quad x=2-c.$

Solut.: $\{(x,y,z) \mid x=2-c,\ y=-1-c,\ z=c,\ c\in\mathbb{R}\}$.

Writing the system in matrix notation.

A system of m equations in n unknowns, over $\mathbb{F}$.

Unknowns: $x_1,\dots,x_n$

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m \end{cases}$$

linear equation

$a_{ij}\in\mathbb{F}$, $1\le i\le m$, $1\le j\le n$, $\quad b_i\in\mathbb{F}$, $1\le i\le m$.

row i

col j

Solutions $x_1,\dots,x_m\in\mathbb{F}$.

Matrix notation.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad m\times n \text{ mat.}$$

Sometimes we write $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

column vect.

(S): $A\cdot x = b$

$L(S) = \{(x_1,\dots,x_n) \mid x_i \in F,\ A\cdot x = b\} \subseteq F^n$

$$\begin{array}{r} \text{Row } 1 \to \\ \\ \text{Row } i \to \\ \\ \text{Row } m \to \end{array} \begin{pmatrix} \boxed{a_{11}\ a_{12} \cdots a_{1n}} \\ \vdots \\ \boxed{a_{i1}\ a_{i2} \cdots a_{in}} \\ \vdots \\ \boxed{a_{m1}\ a_{m2} \cdots a_{mn}} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}\cdot x_1 + a_{12}\cdot x_2 + \dots + a_{1n}\cdot x_n \\ \vdots \\ a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \\ \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \end{pmatrix}$$

So $A\cdot x = b$ is the same as our orig. system of eq.

Exp. $\begin{cases} x + 3y + 4z = -1 \\ 2x - y + z = 5 \end{cases} \iff \begin{cases} x_1 + 3x_2 + 4x_3 = -1 \\ 2x_1 - x_2 + x_3 = 5 \end{cases}$

$$\iff \underset{2\times 3}{\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} -1 \\ 5 \end{pmatrix}}.$$

Goal. study the set/space of solutions of $Ax = b$.

$\exists / \nexists$ solutions? How to find all of them? set of solut. has some struct.?

<u>Notation.</u> We do NOT need to write each time the unknowns $x_1, \ldots, x_n$. Enough to have A & b.

<u>Elementary row operations</u>

Extended matrix $\left(\begin{array}{c|c} A & b \end{array} \right)$

e.g. $\begin{cases} 2x_1 - x_2 + 3x_3 + 2x_4 = 0 \\ x_1 + 4x_2 \qquad - x_4 = 3 \\ 2x_1 + 6x_2 - x_3 + 5x_4 = 9 \end{cases}$ $\qquad \left(\begin{array}{cccc|c} 2 & -1 & 3 & 2 & 0 \\ 1 & 4 & 0 & -1 & 3 \\ 2 & 6 & -1 & 5 & 9 \end{array} \right)$

<u>Elementary row operations.</u>

(1) Choose $0 \neq c \in F$; multiply eq./row i by c. $(c \cdot R_i \longrightarrow R_i)$

(2) choose $c \in F$, $1 \leq i, j \leq m$, $i \neq j$; replace equation/row j by equation/row j <u>plus</u> c <u>times</u> eq./row i.

$(R_j + c \cdot R_i \longrightarrow R_j)$.

(3) Interchange eq./row i and eq./row j. $(R_i \longleftrightarrow R_j)$.

<u>Def.</u> Let C and C' be $r \times s$ matrices with entries in F.

We say that C' is row equivalent to C if C' can be obtained from C be a finite sequence of elementary row operations.

$$C = C_0 \longrightarrow C_1 \longrightarrow \dots \longrightarrow C_k = C'.$$

Thm. Let $\underset{(S)}{A\cdot x = b}$ and $\underset{(S')}{A'\cdot x = b'}$ be two systems of m linear equations in n-unknowns.

Suppose the extended mat. $(A' \vdots b')$ is row equiv. to $(A \vdots b)$. Then $L(S') = L(S)$.

Proof. We begin with the following claim.

Claim 1. If S' is obtained from S by one elementary row operation, then $L(S) \subseteq L(S')$.

Proof of Claim 1. Let $(x_1, \dots, x_n) \in L(S)$. We need to show $(x_1,\dots,x_n) \in L(S')$.

op 1. $c\cdot R_i \to R_i$ $(c \neq 0)$. All the equations of S' coincide with the eqs. of S, except of eq. i

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i \longrightarrow c\cdot a_{i1}x_1 + \dots + c\cdot a_{in}x_n = c\cdot b_i$$

If $(x_1, \dots, x_n)$ satisfies then it satisfies also this

This proves our claim for op. 1.

op. 2. $R_j + c\cdot R_i \longrightarrow R_j$. Again, all the eqs. of S and S' coincide except of eq. j.

$$a_{j1}x_1+\dots+a_{jn}x_n=b_j \longrightarrow a_{j1}x_1+\dots+a_{jn}x_n + c\,a_{i1}x_1+\dots+c\cdot a_{in}x_n = b_j+c\cdot b_i$$

(Row j in S) (Row j in S')

If $(x_1,\dots,x_n)$ satisfies (S) then it satisfies also (S').

<u>Op3.</u> $R_i \longleftrightarrow R_j$. In this case we clearly have:

$$(x_1,\dots,x_n)\in L(S) \Longrightarrow (x_1,\dots,x_n)\in L(S')$$

b.c. the order of the equations in the system does NOT have any effect on the set of solutions.

This concludes the proof of claim 1.

<u>Claim 2.</u> If S' is obtained from S by one element. row operation, then S can be obtained from S' by one elementary row operation.

<u>**Proof of claim 2.**</u> There are 3 cases.

(1) $S \xrightarrow{c\cdot R_i \to R_i} S'$. But then $S' \xrightarrow{R_i \to \frac{1}{c}R_i} S$.

(2) $S \xrightarrow[(i\neq j)]{R_j+c\cdot R_i \to R_j} S'$. But then $S' \xrightarrow{R_j - c\cdot R_i \to R_j} S$.

(3) $S \xrightarrow{R_i \leftrightarrow R_j} S'$. But then $S' \xrightarrow{R_j \leftrightarrow R_i} S$.

This concludes the proof of claim 2.

Claim 3. If S' is obtained from S by one elementary row operation then $L(S)=L(S')$.

Proof of claim 3. By claim 1, $L(S)\subseteq L(S')$. By claim 2, S is obtained from S' by one element. row oper., hence by claim 1 again (with the roles of S and S' reversed) we have $L(S')\subseteq L(S)$. This concludes the proof of claim 3.

We are now in position to prove the Thm.

By assumpt. there is a finite seq. of element. row oper. $S=S_0\longrightarrow S_1\longrightarrow \dots \longrightarrow S_k=S'$.

By claim 3, $L(S)=L(S_0)=L(S_1)=\dots=L(S_k)=L(S')$

$\blacksquare$

Def. An $m \times n$ matrix A is called row-reduced if the following two conditions hold:

(a) The 1'st non-zero entry in each non-zero row of A is 1. (a.k.a pivot/leading term)

(b) Eeach column of A which contains the leading non-zero entry of some row has all its other entries 0.

Examples.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ is row reduced.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ is <u>NOT</u> row reduced

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$ is <u>NOT</u> row reduced

$$D = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \end{pmatrix}$$ is row reduced.

Thm. Every $m\times n$ matrix A is row equivalent to a row-reduced matrix.

Proof. Let $A=(a_{ij})$, $1\leq i\leq m$, $1\leq j\leq n$, $a_{ij}\in F$.

If all entries in the 1'st row of A are 0, then condition (a) is satisfied for row #1.

If row #1 has a non-zero entry, let k be the smallest index j for which $a_{1j}\neq 0$.

(i.e. $a_{1j}=0 \;\; \forall\; 1\leq j<k$ and $a_{1k}\neq 0$).

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1k} & \cdots \\ & & & & \\ & & & & \end{pmatrix}$$

Multiply row #1 by $\frac{1}{a_{1k}}$ and now condition (a) is satisfied for row #1. Now, for each $2\leq i\leq m$ add $(-a_{ik})$ times row #1 to row i

$\left(-a_{ik}\cdot R_1+R_i \longrightarrow R_i\right)$

$$\begin{array}{c} \\ \\ \text{row } i \rightarrow \\ \\ \end{array}
\begin{array}{c} \text{col. } k \\ \downarrow \\ \left(\begin{array}{ccccc} 0 & \cdots & 0 & 1 & \cdots \\ \vdots & & & \vdots & \\ a_{i1} & \cdots & & a_{ik} & \cdots \\ \vdots & & & \vdots & \end{array}\right) \end{array}
\xrightarrow{-a_{ik}\cdot R_1+R_i \longrightarrow R_i}
\begin{array}{c} \text{col. } k-1 \quad \text{col. } k \\ \downarrow \qquad\quad \downarrow \\ \left(\begin{array}{ccccc} 0 & \cdots & 0 & 1 & \cdots \\ \vdots & & \vdots & \vdots & \\ a_{i1} & \cdots & a_{i,k-1} & 0 & \cdots \\ \vdots & & \vdots & \vdots & \end{array}\right) \end{array}
\begin{array}{c} \\ \\ \leftarrow \text{row } i \\ \\ \end{array}$$

We now turn to row #2. If all entries in row #2 are 0 we do nothing to this row. If some entry in row #2 is non-zero we multiply row #2 by a scalar (element from F) so that the leading term (pivot) in row #2 becomes 1. Recall that row #1 had a leading non-zero entry in col. k. It follows that the leading non-zero term in row #2 can NOT be in col. k. It must appear in a different col. say in col. k', where $k' \neq k$. We add suitable multiples of row #2 to all the other rows till we get a matrix in which all entries in col. k' are 0, except the one in row #2.

$$\begin{array}{c}\begin{array}{cccccccc} & & k' & & k & & & \end{array}\\ \left(\begin{array}{cccccccc} 0 & \cdots & 0 & \cdots 0 & 1 & * & \cdots & * \\ 0 & \cdots 0 & 1 & * \cdots * & 0 & * & \cdots & * \\ & & 0 & & 0 & & & \\ & & 0 & & 0 & & & \\ & & \vdots & & \vdots & & & \\ & & 0 & & 0 & & & \end{array}\right)\end{array} \quad \text{or} \quad \begin{array}{c}\begin{array}{cccccccc} & & k & & k' & & & \end{array}\\ \left(\begin{array}{cccccccc} 0 & \cdots & 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots & * \\ 0 & \cdots & 0 \cdots 0 & 0 & \cdots 0 & 1 & * \cdots & * \\ & & & 0 & & 0 & & \\ & & & \vdots & & \vdots & & \\ & & & 0 & & 0 & & \end{array}\right)\end{array}$$

__Important point.__ In the last operations using row#2 we do NOT change any of the entries in row#1 that are in cols. $1, \dots, k$. Also nothing will be changed in column k.

Also, in case row#1 was 0, the operations using row#2 will not affect row#1.

We continue now to row #3 and do the same procedure. After a finite number of elementary row operations we arrive at a row-reduced matrix

$\blacksquare$

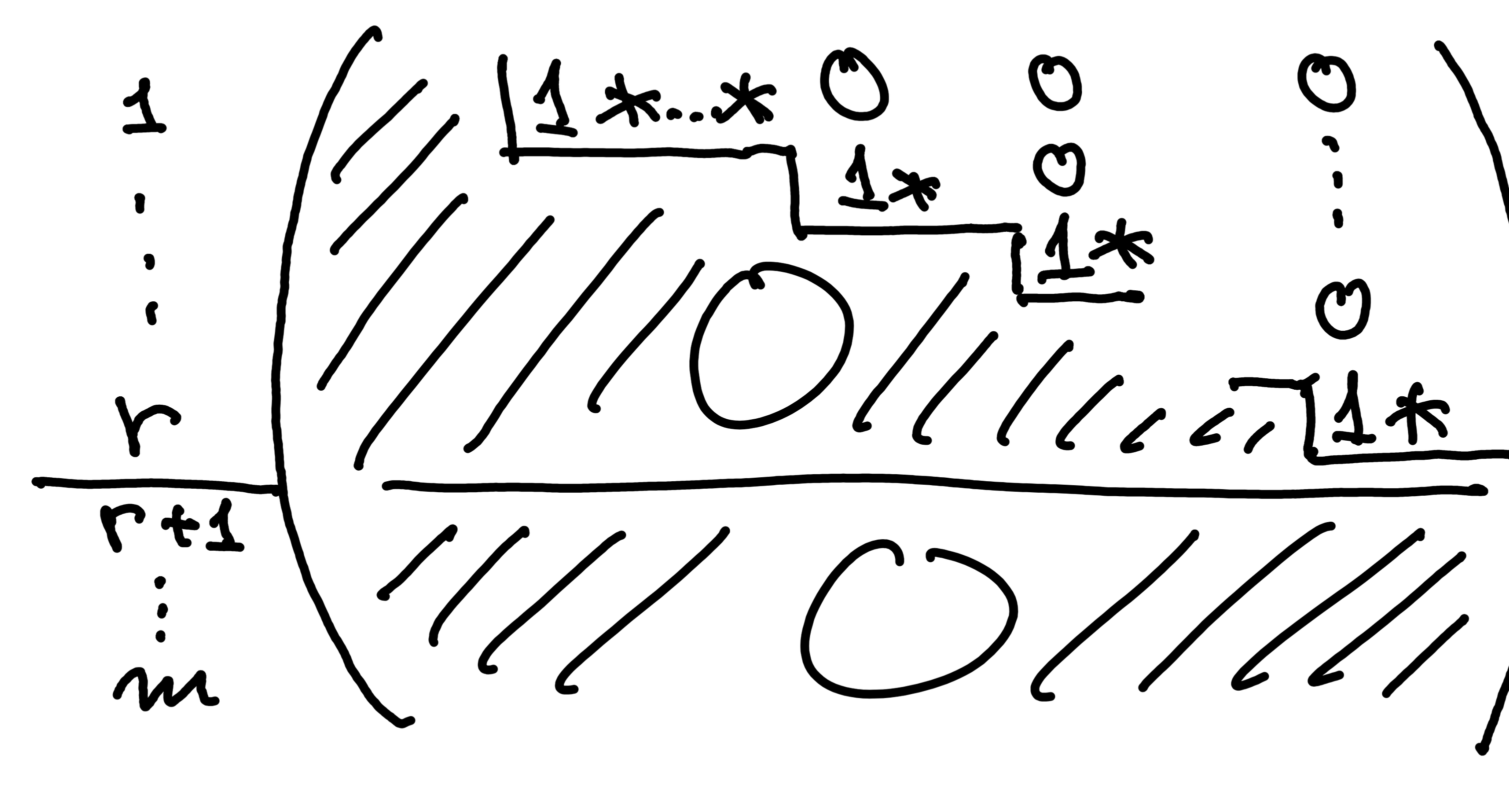
Def. An $m \times n$ matrix A is called row-reduced echelon if the following holds:

(a) A is row-reduced.

(b) Every row of A which has all its entries 0 appears below every row which has a non-zero entry.

(c) If rows $1, \dots, r$ are the non-zero rows of A and if the leading non-zero entry of row i occurs in column k_i, $i = 1, \dots, r$, then

$k_1 < k_2 < \dots < k_r$.



Thm. Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Proof. We already know that A is equiv. to a row-red. matrix A'. After performing a finite number of row interchanges on A' we arrive at a row-red. echel. mat. ▨

Why is this usefull for solving systems of eqs.?

Remark. Let $Ax=b$ be a system of lin. eqs.

Consider $(A \,\vdots\, b)$ the extended matrix.

We can do the same elementary operations that we do to A also to $(A \,\vdots\, b)$ and obtain a row-equiv. matrix $(A' \,\vdots\, b')$ where A' is a row-reduced echelon mat. The systems $Ax=b$ and $A'x=b'$ will have the same solutions but $A'x=b'$ is easier to solve.

Example.

$$\left(\begin{array}{cccc|c} 0 & -9 & 3 & 4 & 9 \\ 1 & 4 & 0 & -1 & 5 \\ 2 & 6 & -1 & 5 & -5 \end{array}\right) \xrightarrow{-\frac{1}{9}R_1 \to R_1} \left(\begin{array}{cccc|c} 0 & 1 & -\frac{1}{3} & -\frac{4}{9} & -1 \\ 1 & 4 & 0 & -1 & 5 \\ 2 & 6 & -1 & 5 & -5 \end{array}\right)$$

$$\xrightarrow[R_3 - 6R_1 \to R_3]{R_2 - 4R_1 \to R_2} \left(\begin{array}{cccc|c} 0 & 1 & -\frac{1}{3} & -\frac{4}{9} & -1 \\ 1 & 0 & \frac{4}{3} & \frac{7}{9} & 9 \\ 2 & 0 & 1 & \frac{23}{3} & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left(\begin{array}{cccc|c} 0 & 1 & -\frac{1}{3} & -\frac{4}{9} & -1 \\ 1 & 0 & \frac{4}{3} & \frac{7}{9} & 9 \\ 0 & 0 & -\frac{5}{3} & \frac{55}{9} & -17 \end{array}\right)$$

$$\xrightarrow{-\frac{3}{5}R_3 \to R_3} \left(\begin{array}{cccc|c} 0 & 1 & -\frac{1}{3} & -\frac{4}{9} & -1 \\ 1 & 0 & \frac{4}{3} & \frac{7}{9} & 9 \\ 0 & 0 & 1 & -\frac{11}{3} & \frac{51}{5} \end{array}\right)$$

$$\xrightarrow[R_1 + \frac{1}{3}R_3 \to R_1]{R_2 - \frac{4}{3}R_3 \to R_2} \left(\begin{array}{cccc|c} 0 & 1 & 0 & -\frac{5}{3} & \frac{12}{5} \\ 1 & 0 & 0 & \frac{17}{3} & -\frac{23}{5} \\ 0 & 0 & 1 & -\frac{11}{3} & \frac{51}{5} \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{17}{3} & \frac{-23}{5} \\ 0 & 1 & 0 & \frac{-5}{3} & \frac{12}{5} \\ 0 & 0 & 1 & \frac{-11}{3} & \frac{51}{5} \end{array}\right).$$

The corresponding system:

$$\begin{cases} x_1 + \frac{17}{3}x_4 = \frac{-23}{5} \\ x_2 - \frac{5}{3}x_4 = \frac{12}{5} \\ x_3 - \frac{11}{3}x_4 = \frac{51}{5} \end{cases}$$

Take $x_4 := a \in \mathbb{R}$ arbitrary.

Then $x_3 = \frac{11}{3}a + \frac{51}{5}$, $x_2 = \frac{5}{3}a + \frac{12}{5}$, $x_1 = \frac{-17}{3}a - \frac{23}{5}$

And these are all the solutions.

Another example.

Fix $b_1, b_2, b_3 \in \mathbb{R}$

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Let's see how the solutions behave for a general choice of b_1, b_2, b_3.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & b_1 \\ 2 & 1 & 1 & b_2 \\ 0 & 5 & -1 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & b_1 \\ 0 & 5 & -1 & b_2 - 2b_1 \\ 0 & 5 & -1 & b_3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & b_1 \\ 0 & 5 & -1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & b_1 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(b_2 - 2b_1) \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{1}{5}(b_1 + 2b_2) \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(b_2 - 2b_1) \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right)$$

So the equivalent system is:

$$\begin{cases} x_1 \qquad + \frac{3}{5}x_3 = \frac{1}{5}(b_1+2b_2) \\ \quad x_2 - \frac{1}{5}x_3 = \frac{1}{5}(b_2-2b_1) \\ \qquad\qquad 0 = b_3-b_2+2b_1 \end{cases}$$

If $b_3-b_2+2b_1 \neq 0$, then the system has NO solutions.

If $b_3-b_2+2b_1=0$, then the system is equiv.

to $$\begin{cases} x_1 \qquad + \frac{3}{5}x_3 = \frac{1}{5}(b_1+2b_2) \\ \quad x_2 - \frac{1}{5}x_3 = \frac{1}{5}(b_2-2b_1) \end{cases}$$

Take $x_3 = c \in \mathbb{R}$ arbitrary.

$$x_2 = \frac{1}{5}c + \frac{1}{5}(b_2-2b_1), \quad x_1 = \frac{-3}{5}c + \frac{1}{5}(b_1+2b_2).$$