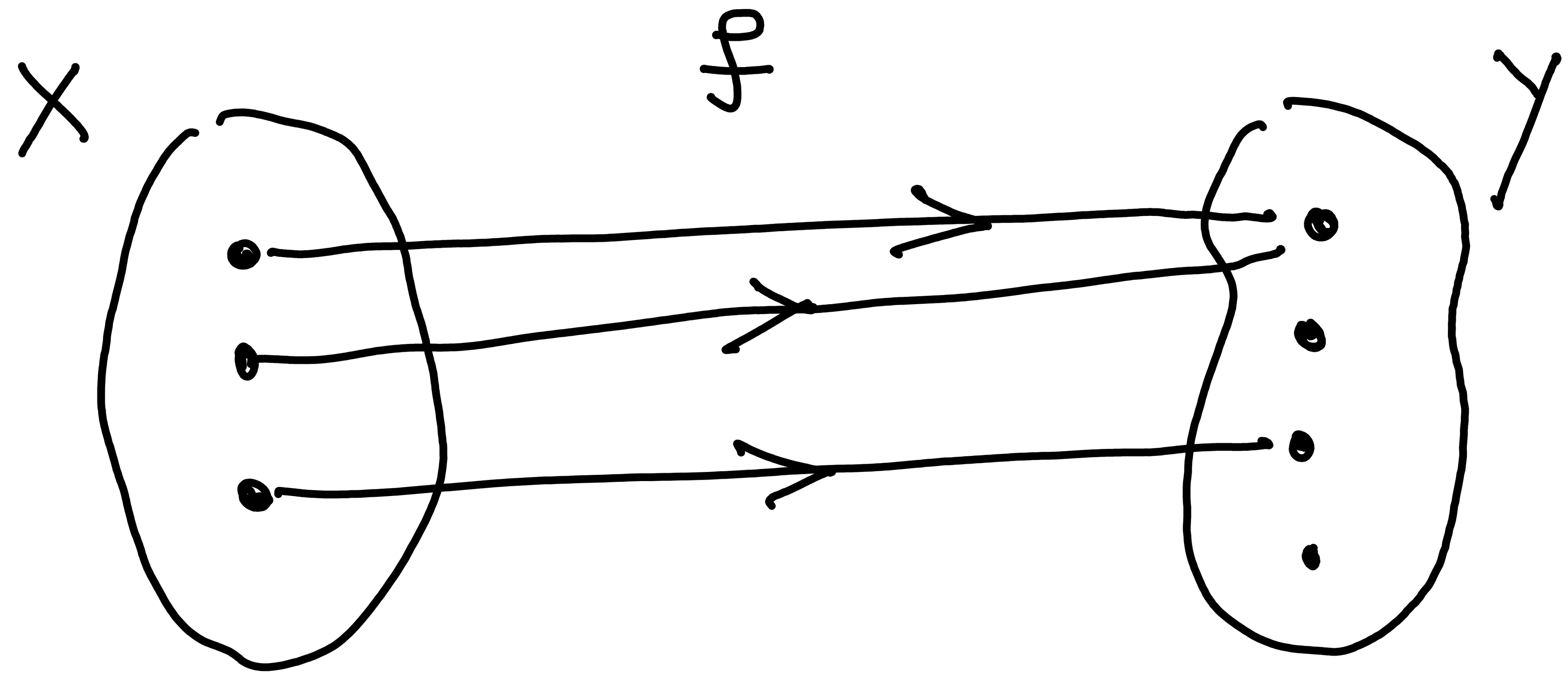
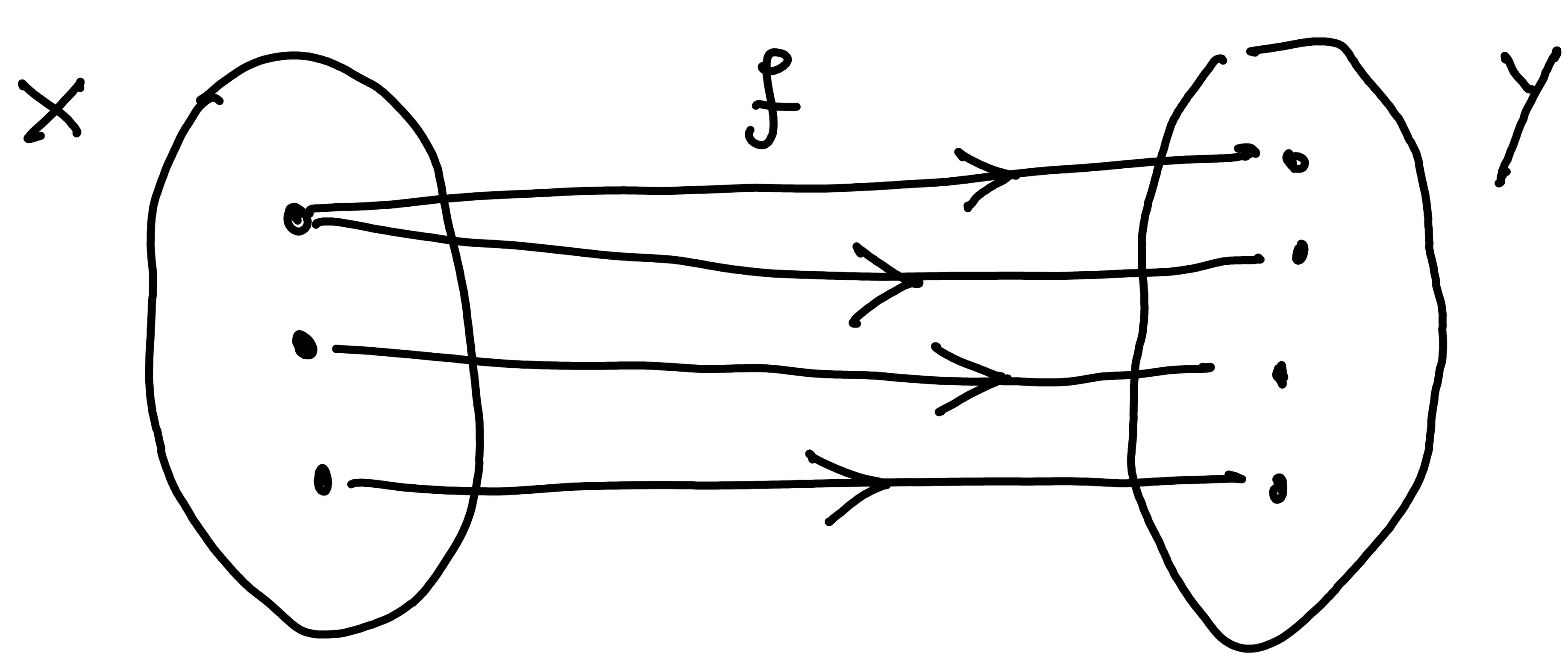
Maps.

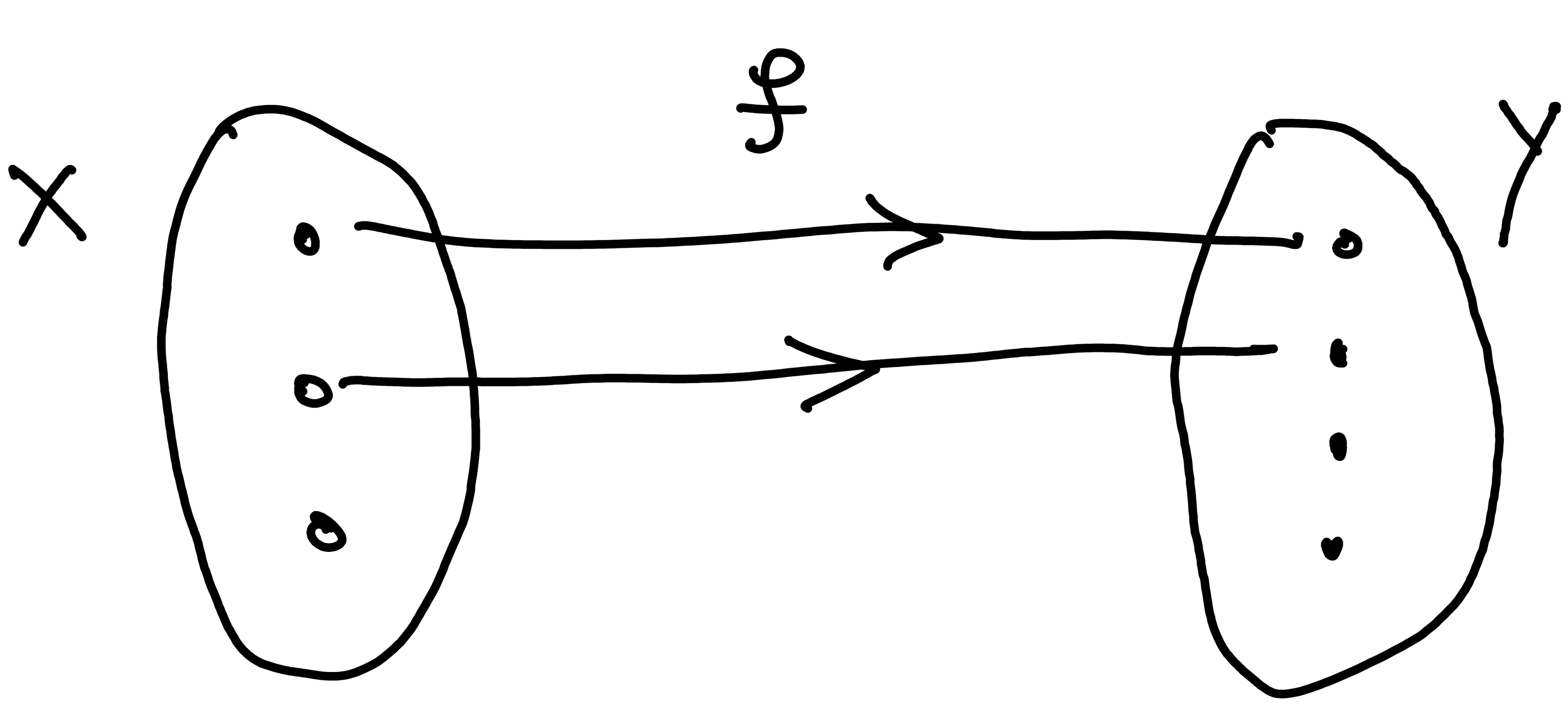
Def. Let X, Y be sets. A map $f : X \longrightarrow Y$ (also function, transformation) is an assignment that assigns to <u>every</u> element $x \in X$ a <u>uniquely defined</u> element $f(x) \in Y$.



O.K.



<u>NOT</u> O.K.



NOT O.K.

Input $\in X \xrightarrow{f}$ output $\in Y$

X is called the domain of definition of f.

Y is called the target (or target domain) of f.

Two functions $f: X \to Y$, $g: X' \to Y'$ are equal if they have the same domain of def., $X' = X$, the same target, $Y' = Y$, and also $\forall\ x \in X$ we have $f(x) = g(x)$.

Sometimes a funct. is given by a formula, sometimes not.

$f(x) = x^2$ (Bad! What's the domain/target?)

$X := \{x \in \mathbb{R} \mid 0 \leq x\}$, $f: X \to \mathbb{R}$, $f(x) = x^2$ (o.k.)

$X = \{x \in \mathbb{Z} : x \geq 0\}$, Y = set of family names

$g: X \to Y$, $g(x) :=$ family name of student at ETH whose stud. # is x.

g is not defined..., $f(0) = ?$, $f(10^{10}) = ?$

$\mathcal{Y}'$ = set of family names $\cup$ {ERROR!}

$$h(x) = \begin{cases} \text{family name of stud. } \# x & \text{if } \exists \text{ a student with student no. } x \\ \text{ERROR!} & \text{otherwise} \end{cases}$$

Notation: $f : X \to \mathcal{Y}$, $X \ni x \longmapsto f(x) \in \mathcal{Y}$.

E.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$, $\mathbb{R} \ni x \xmapsto{f} x^5 \in \mathbb{R}$.

<u>Def.</u> Let $f : X \to \mathcal{Y}$ be a map. The image of f is the set $f(X) := \{ y \in \mathcal{Y} \mid \exists\, x \in X \text{ s.t. } f(x) = y \}$.

(sometimes $\text{image}(f)$). $f(X) \subseteq \mathcal{Y}$.

<u>Def.</u> Let $f : X \to \mathcal{Y}$ be a map. Let $A \subseteq X$ be a subset. We can define a new map

$f|_A : A \longrightarrow \mathcal{Y}$, the restriction of f to A,

just by $f|_A(x) := f(x) \quad \forall\, x \in A$.

f is also called an extension of $f|_A$.

Exps. * $id : X \longrightarrow X$, $x \longmapsto x$ identity map.

(sometimes id_X).

* The constant function. $f : X \longrightarrow Y$, $y_0 \in Y$.

$f(x) := y_0 \quad \forall\, x \in X$.

* X set. $A \subseteq X$. The characteristic function on A: $\mathbb{1}_A : X \longrightarrow \mathbb{R}$, $\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$

* X_1, X_2 sets. Consider $X_1 \times X_2$.

We have two projection maps

$$\pi_1 : X_1 \times X_2 \longrightarrow X_1, \quad (x_1, x_2) \longmapsto x_1$$

$$\pi_2 : X_1 \times X_2 \longrightarrow X_2, \quad (x_1, x_2) \longmapsto x_2.$$

Def. Let $f: X \to Y$ be a map.

1) We say that f is injective (or: f is an injection) if: $\forall\, x_1, x_2 \in X$ s.t. $f(x_1) = f(x_2)$ we have $x_1 = x_2$. $\left(\forall\, x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2\right)$

(Alternatively: $\forall\, x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$).

2) We say f is surjective (or: f is a surjection) if $\forall\, y \in Y$, $\exists\, x \in X$ s.t. $f(x) = y$.

(equiv.: $f(X) = Y$).

3) We say f is bijective (or: f is a bijection) if f is both inject. & surject.

(equiv.: $\forall\, y \in Y, \exists!\, x \in X : f(x) = y$).

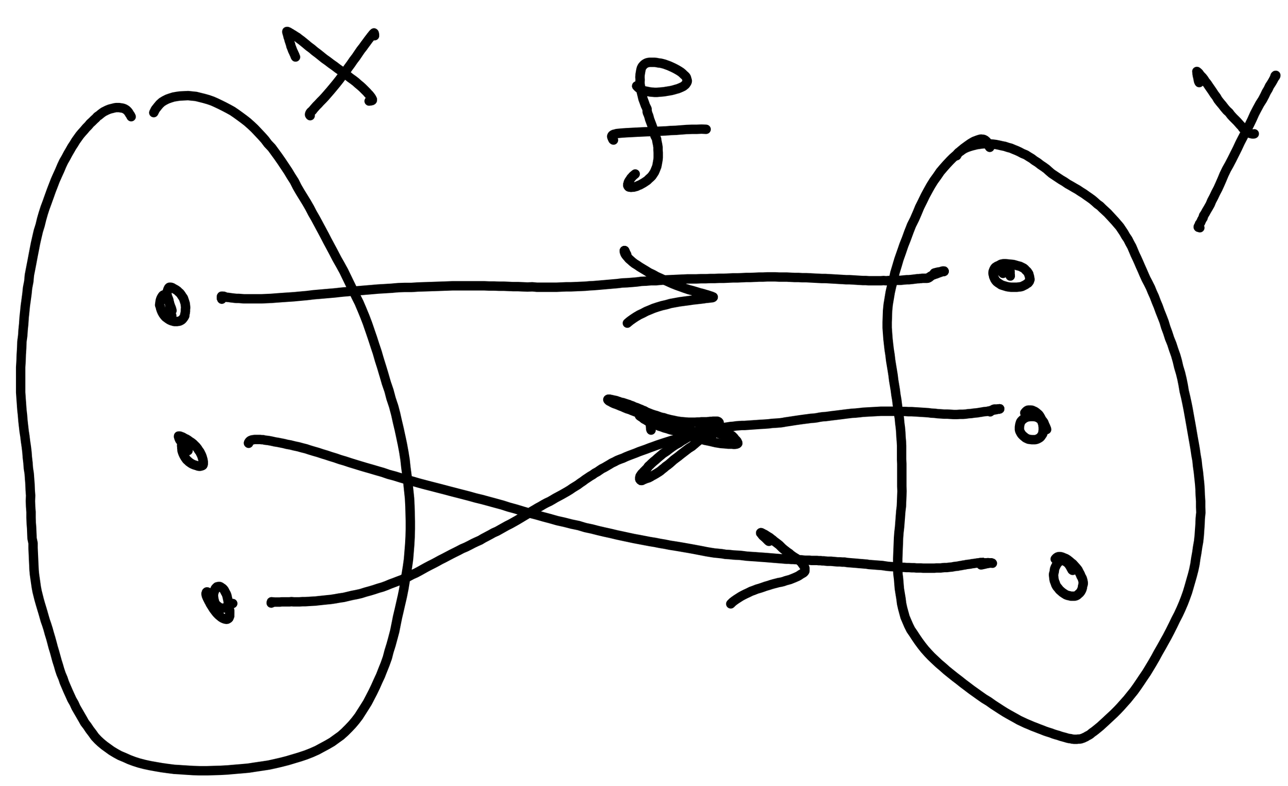
Exp. * $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Not inj. not surj.

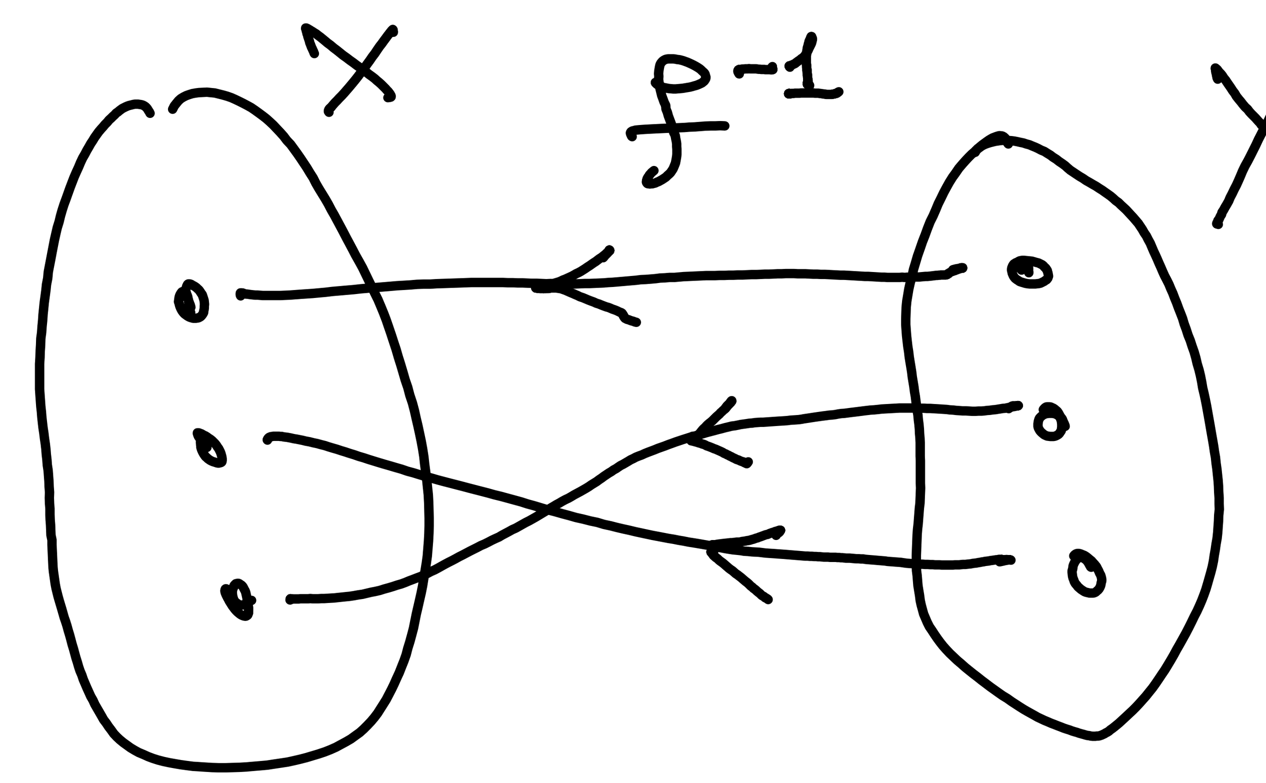
* $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) = x^2$. inj. but not surj.

* $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f(x) = x^2$. biject.

Def. Let $f: X \to Y$ be a bijection.

The inverse map $f^{-1}: Y \to X$ is the map that assigns to every $y \in Y$ the unique $x \in X$ s.t. $f(x) = y$.





Exp. $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f(x) = x^2$

$f^{-1}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f^{-1}(y) = \sqrt{y}$.

Composition.

Def. Let $f: X \to Y$, $g: Y \to Z$ be maps.

$g \circ f: X \to Z$, $g \circ f(x) := g(f(x))$.

$$x \xmapsto{f} f(x) \xmapsto{g} g(f(x))$$

Exp. $f: X \to Y$ biject. $f^{-1}: Y \to X$ inverse

$f^{-1} \circ f = id_X$, $f \circ f^{-1} = id_Y$.

<u>Remarks.</u> $g \circ f$ is defined, but $f \circ g$ in general is NOT defined (unless $Z = X$).

But even when $Z = X$, <u>in general</u> $g \circ f \neq f \circ g$.

<u>Exp.</u>
$$\left.\begin{array}{l} f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x+1 \\ \\ g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2 \end{array}\right\} \begin{array}{l} g \circ f : \quad x \mapsto x+1 \mapsto (x+1)^2 \\ \qquad\qquad\qquad\qquad = x^2+2x+1 \\ f \circ g : x \mapsto x^2 \mapsto x^2+1 \quad \neq \end{array}$$

<u>Lemma.</u> Let $f : X \to Y$, $g : Y \to Z$ be maps.

1) If f & g are inject. then $g \circ f$ is also inj.

2) If f & g are surj. then $g \circ f$ is also surj.

3) If f & g are bij. then $g \circ f$ is also bij.

<u>Proof.</u> 1) We need to prove that $\forall\, x_1, x_2 \in X$ the following holds: if $g \circ f(x_1) = g \circ f(x_2)$ then $x_1 = x_2$.

So let $x_1, x_2 \in X$ be s.t. $g \circ f(x_1) = g \circ f(x_2)$.

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$. Since g is injective it follows that $f(x_1) = f(x_2)$.

Since f is injective it follows that $x_1 = x_2$.

2) We need to show that $\forall\, z \in Z, \exists\, x \in X$

s.t. $g \circ f(x) = z$.

Indeed, let $z \in Z$. Since g is surjective,

$\exists\, y \in Y$ s.t. $g(y) = z$. Since f is injective,

it follows that $\exists\, x \in X$ s.t. $f(x) = y$.

We now have: $g \circ f(x) = g(f(x)) = g(y) = z$.

3) This follows from 1+2, b.c.

surjective $=$ injective & surjective.

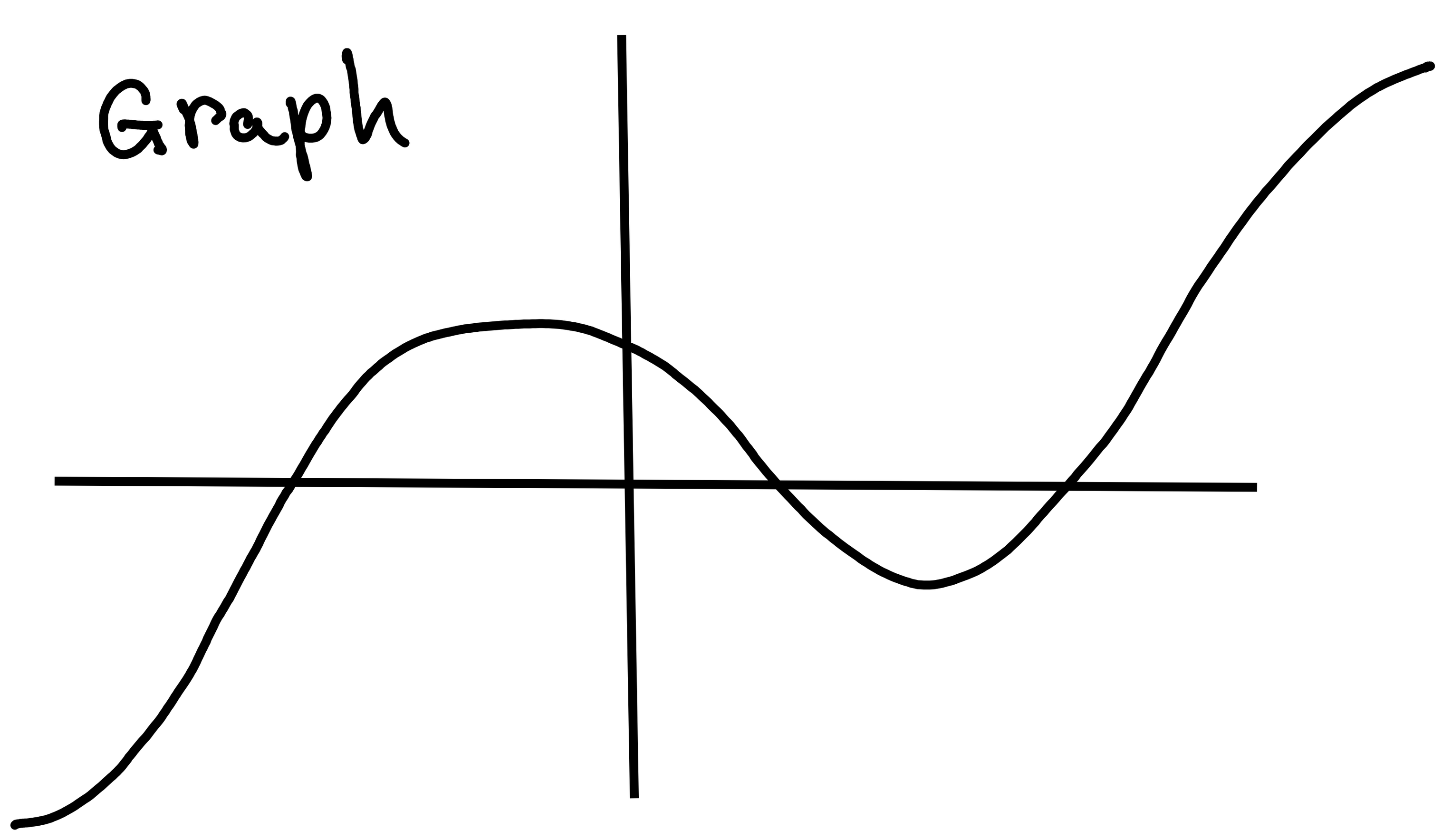
$\blacksquare$

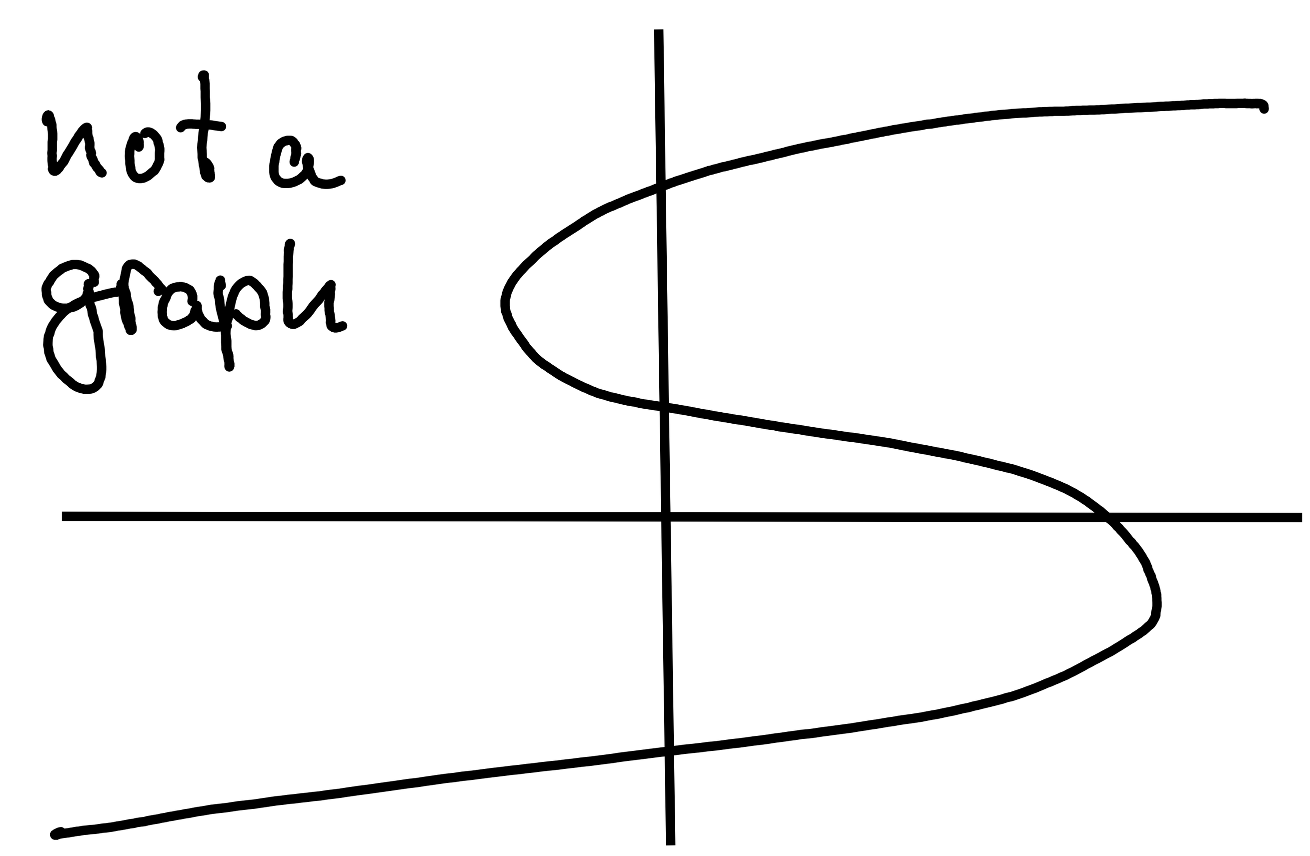
Graphs.

Def. Let $f: X \to Y$ be a map. The graph of f is the following subset $\mathrm{Graph}(f) \subseteq X \times Y$:

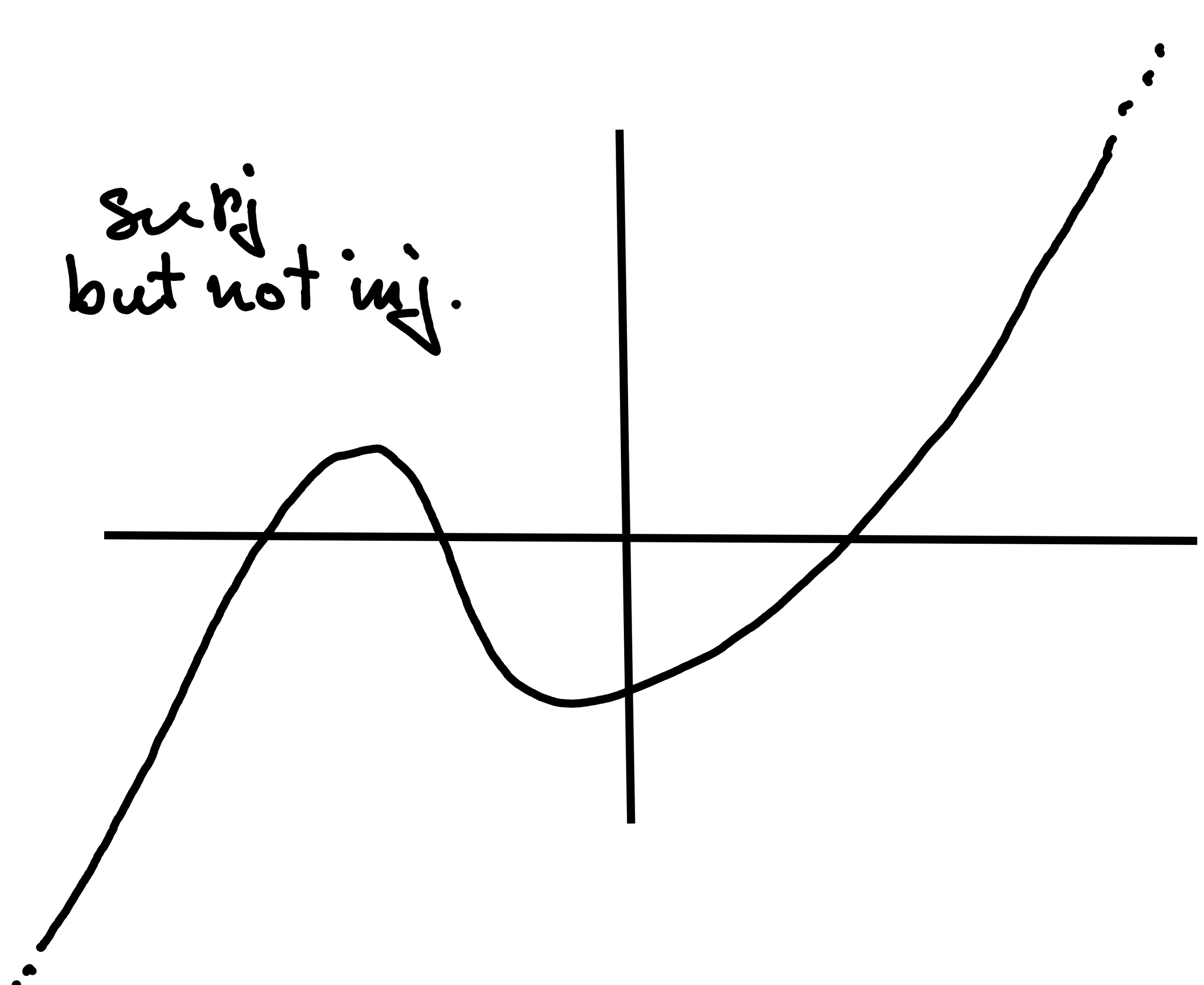
$$\mathrm{Graph}(f) := \{(x,y) \in X \times Y \mid y = f(x)\} \subseteq X \times Y.$$

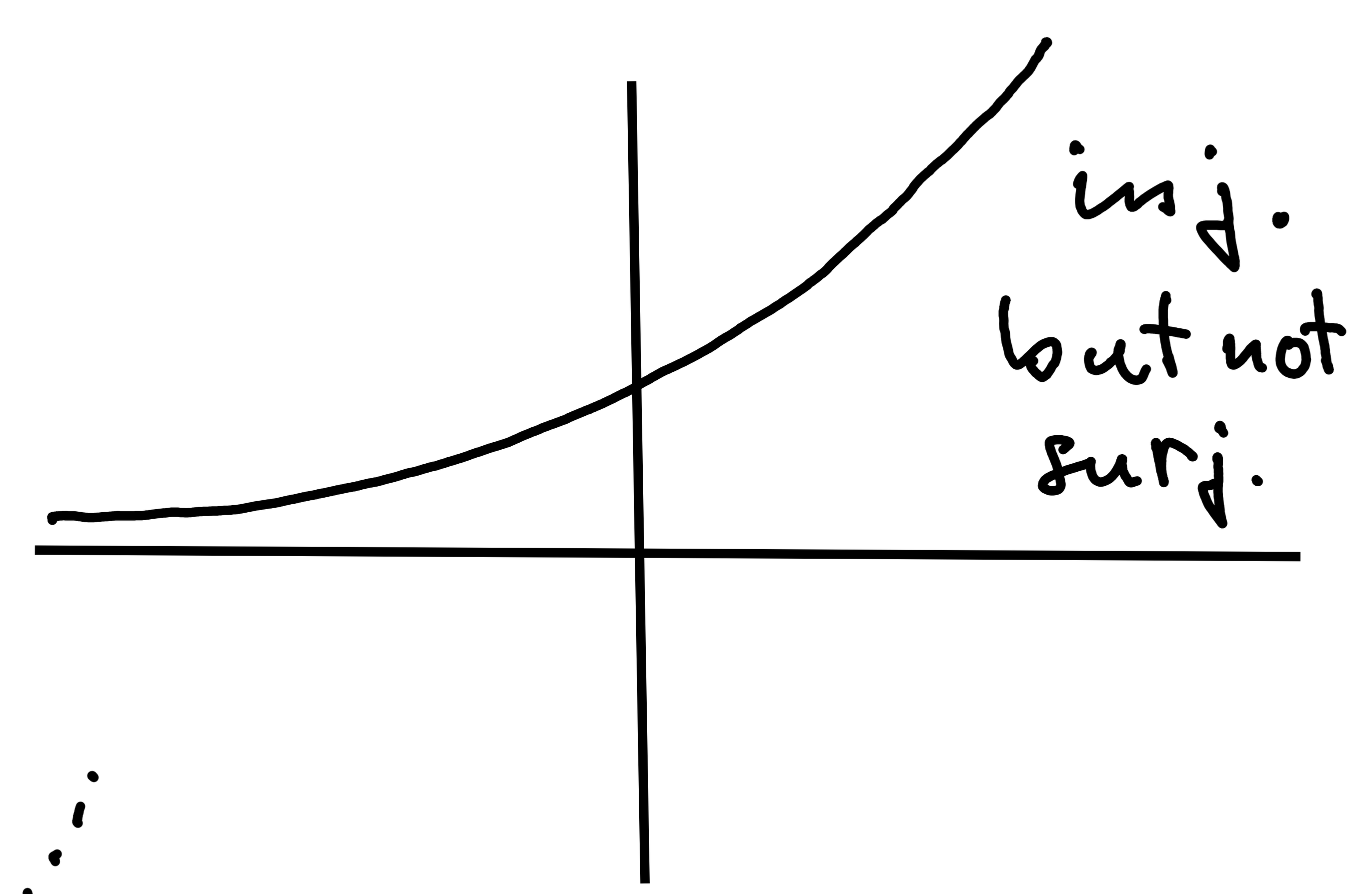
Examples. $X \subseteq \mathbb{R}$, $Y = \mathbb{R}$





Sometimes we can see inj/surj./bij. from the graph:





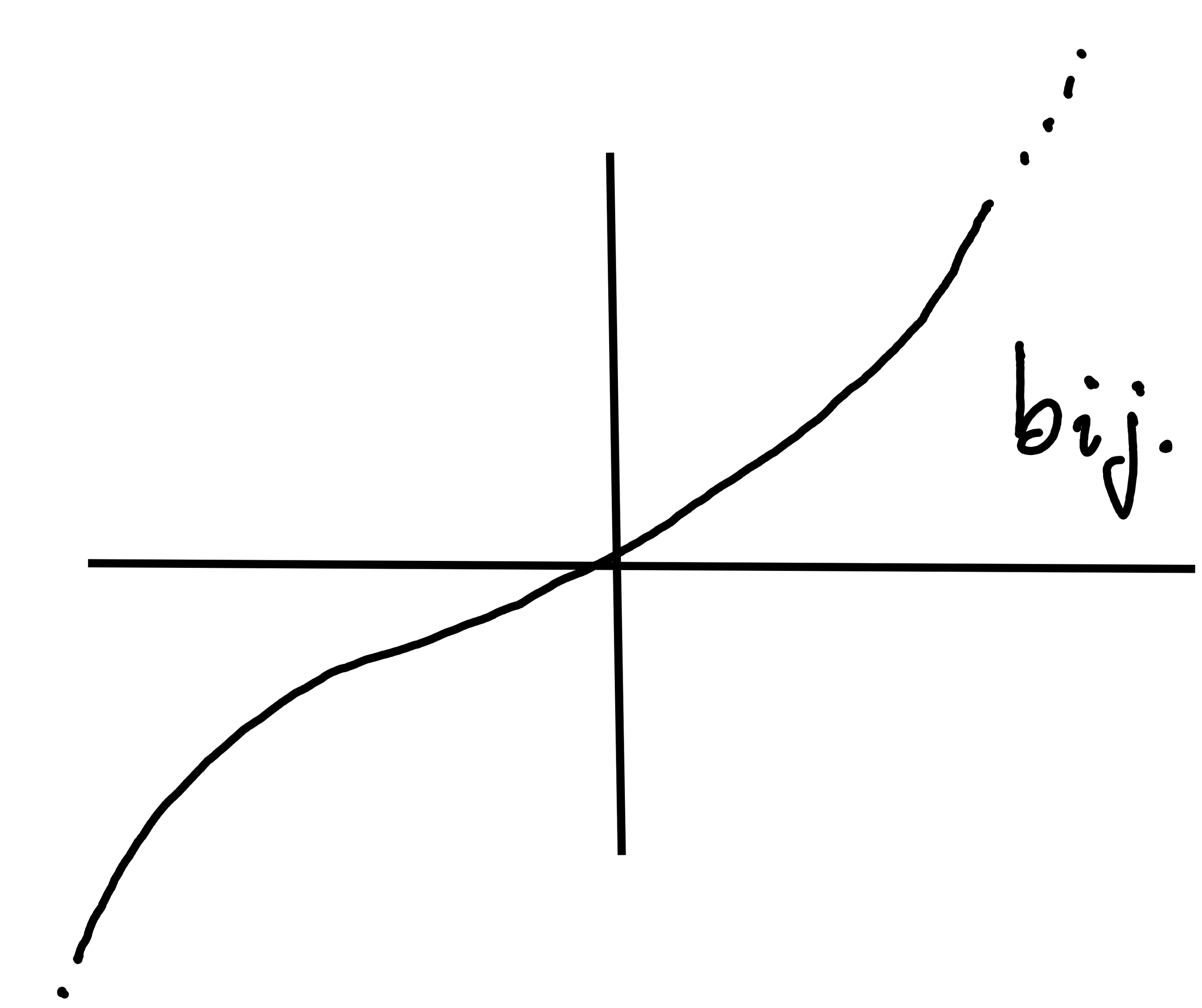


Image and inverse image.

Def. Let $f: X \to Y$ be a map. Let $A \subseteq X$.

$$f(A) := \{y \in Y \mid \exists\, x \in A \text{ s.t. } f(x) = y\}$$
$$= \{f(x) \mid x \in A\}.$$ Image of A under f.

Let $B \subseteq Y$. Define the inverse image of B under f:

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X.$$

confusing notation

Exp. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. (f is not inj., not surj.)

$$f^{-1}(\{4\}) = \{-2, 2\}$$

$$f^{-1}\{1, 9\} = \{-1, 1, -3, 3\}$$

$$f^{-1}(\{0\}) = \{0\}$$

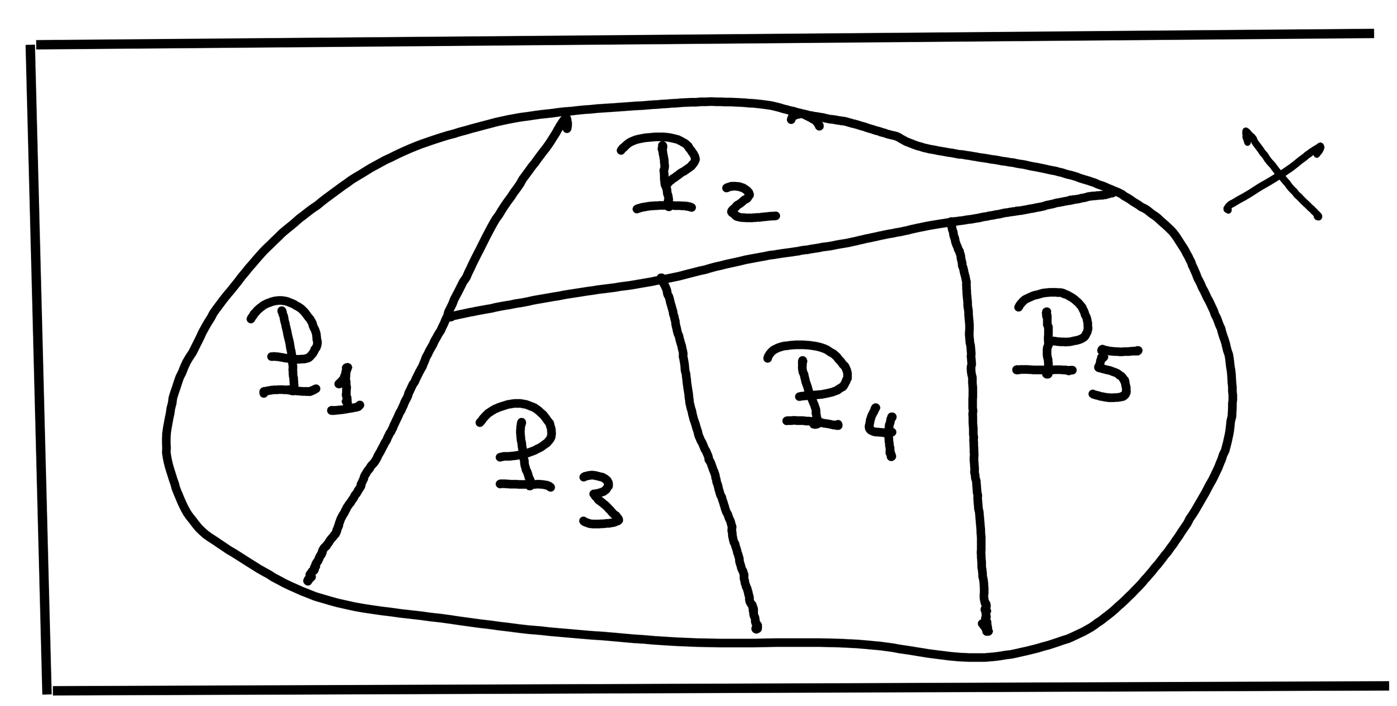
$$f^{-1}(\{-1\}) = \emptyset.$$

$$f^{-1}(\{-1, 0\}) = \{0\}.$$

Partitions.

Def. Let X be a set and $\mathcal{P}$ a family of subsets of X, none of which is empty. We say that $\mathcal{P}$ is a partition of X if the following holds:

1) $X = \bigcup_{P \in \mathcal{P}} P$.
2) $\forall\ P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$, we have that $P_1 \cap P_2 = \emptyset$.



Defining functions by cases.

Let X, Y be sets and $\mathcal{P}$ a partitions of X.

Suppose we are given for every $P \in \mathcal{P}$ a function $f_P : P \longrightarrow Y$. Then we can define a function

$f : X \longrightarrow Y$ with the property that $f|_P = f$ $\forall P \in \mathcal{P}$.

(In fact $\exists!$ a funct. f with this property.)

How? Let $x \in X$. By def. of partit. $\exists!\ P \in \mathcal{P}$ s.t. $x \in P$. Define $f(x) := f_P(x)$.