Set Theory.

A set is a collection of different objects. These objects are called elements of the set.

<u>Exp.</u> $M = \{1, 3, \text{moon}\} = \{3, \text{moon}, 1\}$ (order NOT important).

$M = \{1, 1, 1\} = \{1\}$ (repetitions do NOT matter)

If x is an element of M we write $x \in M$,

If x is NOT ——— " ——— $x \notin M$.

$M = \{x : A(x)\}$ = set of all x that satisfy
$= \{x \mid A(x)\}$ the condition/property $A(x)$.

<u>Exp.</u> $\{x \mid x \text{ is an integer and } x^2 \leq 9\}$

$= \{-3, -2, -1, 0, 1, 2, 3\}$.

<u>Empty set:</u> $\{\} = \phi$ (contains no elements).

A set is allowed to contain another set as one of its elements.

$M = \{1, \text{elephant}, \{a,5\}\}$

$\{a,5\} \in M$. a set! But also an element of M.

$M = \{\{\}\} = \{\emptyset\}$ a set containing exactly one element: the empty set.

Sometimes $P \subset Q$

<u>Def.</u> 1. $P \subseteq Q$ means that $\forall x \in P : x \in Q$.

(or: $x \in P \Rightarrow x \in Q$). We say P is a subset of Q.

2. $P \subsetneq Q$ means $P \subseteq Q$ but $P \neq Q$

3. $P \nsubseteq Q$ means $\neg (P \subseteq Q)$.

<u>Exp.</u> $\{-1, 2, 5\} \subseteq \{x \mid x \text{ is integer and } x^2 \leq 30\}$
$\subsetneq$

$\{-1, 2, 5\} \nsubseteq \{x \mid x \text{ is integer and } x^2 \leq 20\}$.

<u>Operations on sets.</u>

<u>Def.</u> $P \cap Q := \{x \mid x \in P \wedge x \in Q\}$ intersection.

$P \cup Q := \{x \mid x \in P \vee x \in Q\}$ union

$P \setminus Q := \{x \mid x \in P \wedge x \notin Q\}$ difference

$P \triangle Q := (P \cup Q) \setminus (P \cap Q)$.

If our sets (in some discussion) are subsets of one ambient / underlying / Grundmenge X, then we write $P^c := X \setminus P$ the complement of P (in X).

sometimes $\overline{P}$

<u>Def.</u> Let $\mathcal{A}$ be a family (= set) of sets, $\mathcal{A} \neq \emptyset$,

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in \mathcal{A} \text{ s.t. } x \in A\} \quad \text{union}$$

$$\bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\} \quad \text{intersection}$$

In practice, sometimes $\mathcal{A}$ just parametrizes / indexes a family of sets.

<u>Exp.</u> $\mathcal{A} = \{2, 3, \dots\}$.

$\forall n$, $A_n = \{x \mid x \text{ is integer, } x \geq 1 \text{, and } n^2 \text{ divides } x\}$

$A_2 = \{4, 8, 12, \dots\}$, $A_3 = \{9, 18, \dots\}$, ...

$\bigcup_{n \in \mathcal{A}} A_n = \{a \in \mathbb{Z} \mid \exists\, k, r \in \mathbb{Z},\, k, r \geq 1, \text{ s.t. } a = k^2 \cdot r\}$, $\bigcap_{n \in \mathcal{A}} A_n = \emptyset$.

Cartesian product.

Def. Let X, Y be sets. The cartesian product of X and Y, denoted $X \times Y$ is the set of all <u>ordered</u> pairs (x,y) with $x \in X$, $y \in Y$.

Exp. $\underbrace{\{1, 2, \text{elephant}\}}_{X} \times \underbrace{\{4, \text{dog}\}}_{Y} =$

$= \{(1,4), (1,\text{dog}), (2,4), (2,\text{dog}), (\text{elephant},4), (\text{eleph.},\text{dog})\}$

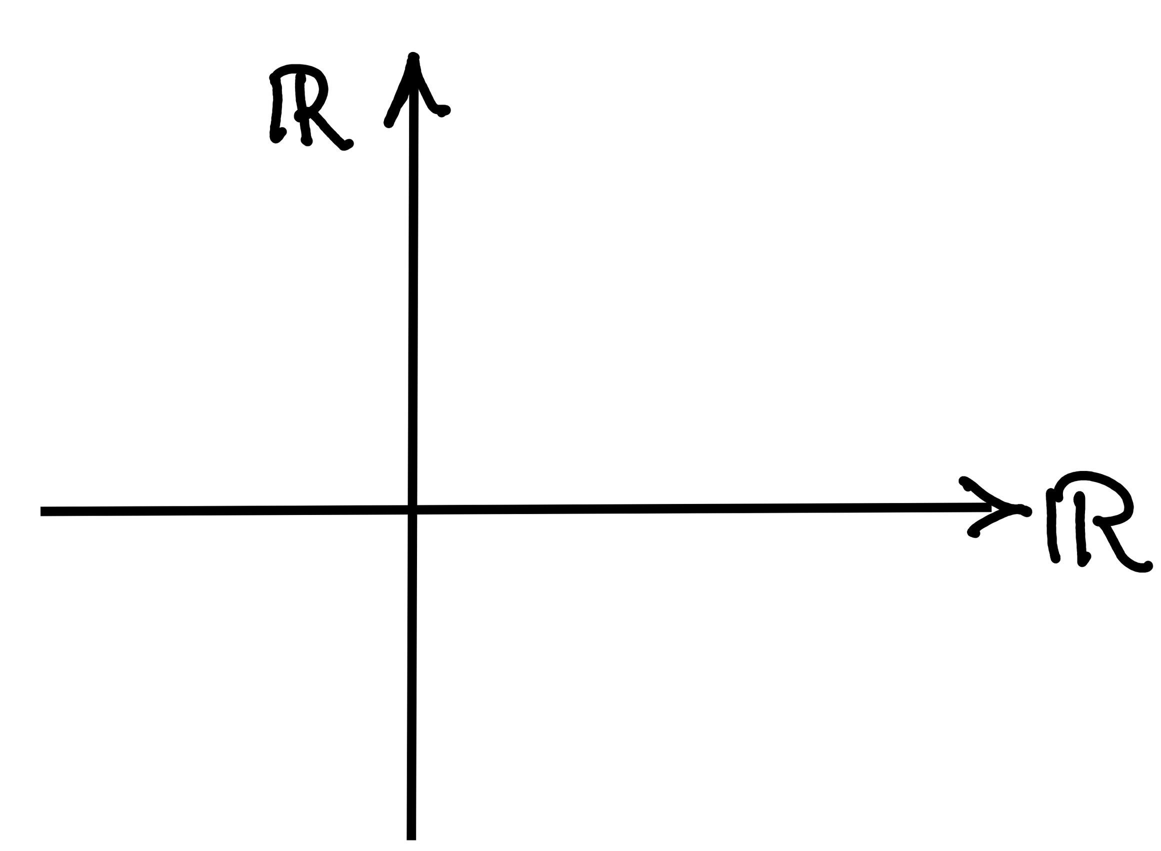
So $(1,4) \in X \times Y$ but $(4,1) \notin X \times Y$.

Def. Let X be a set. We define its n-fold cart. prod

$$X^n = \underbrace{X \times \ldots \times X}_{n \text{ times}} = \{\underbrace{(x_1, \ldots, x_n)}_{n\text{-tuples}} \mid x_1, \ldots, x_n \in X\}$$

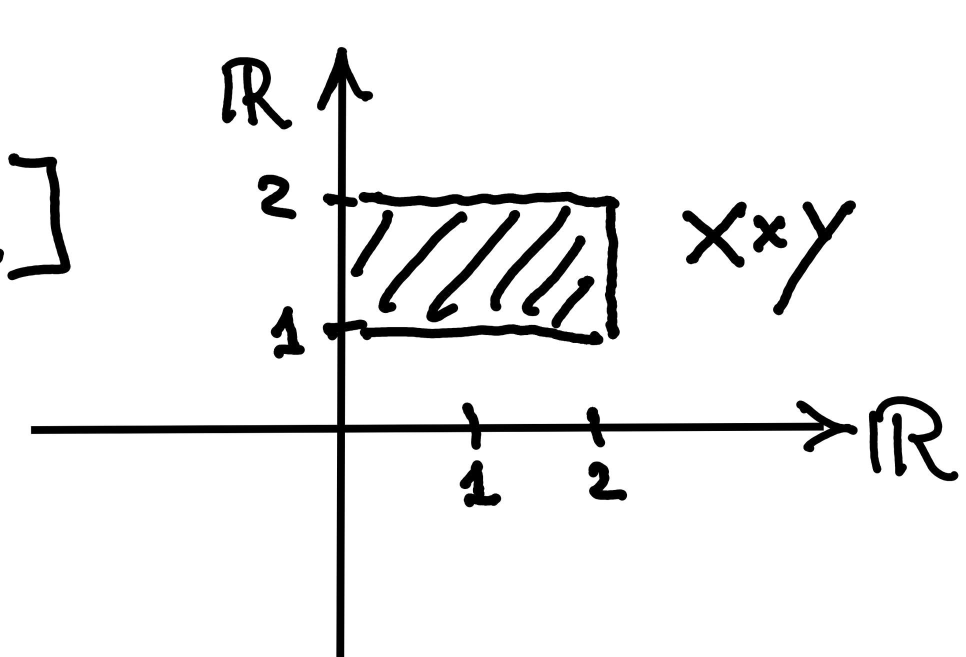
$X^2 = X \times X$. $X^3 = X \times X \times X = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in X\}$, etc.

Exp. $X = \mathbb{R}$ the real numbers



$X^2 = \mathbb{R}^2$ the plane.

$X = [0,2]$, $Y = [1,2]$



Power sets.

Def. Let X be a set.

$$\mathcal{P}(X) := \{ Q \mid Q \text{ is a subset of } X \} \quad (\text{sometimes } 2^X).$$

$$\mathcal{P}(\{1,2,3\}) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}.$$

8 elements.

Cardinality of a set

Let X be a finite set, i.e. a set with finitely many elements; $X = \{x_1, \dots, x_n\}$.

We say that the cardinality of X is n (number of elements in X) and write $|X| = n$ or $\text{card}(X) = n$.

Exercise: Let A be a finite set. Let $\mathcal{P}(A)$ be the power set of A. Show that $|\mathcal{P}(A)| = 2^{|A|}$.

Russel paradox.

$\mathcal{U} :=$ the set of all sets.

$R := \{P \mid P \in \mathcal{U} \text{ and } P \notin P\}$

$=$ the set of all sets that do not contain themselves.

For example: $\mathcal{U} \in \mathcal{U}$. So $\mathcal{U} \notin R$.

Claim. $R \in R \iff R \notin R$.

Proof. If $R \in R$, then by def. of R we have $R \notin R$.

This shows $R \in R \Rightarrow R \notin R$

If $R \notin R$ then by def R does contain itself,

so $R \in R$. This shows $R \in R \Leftarrow R \notin R$.

We will not solve in this course this paradox.

The solution comes in axiomatic set theory (in contrast to "naive set theory"), where one defines more precisely what is a set. Turns out that our $\mathcal{U}$ and R cannot be regarded as sets.