

Vector spaces.

Def. A vector space over a field K is a set V ,
 endowed with two operations:

$$+ : V \times V \rightarrow V, \quad (v_1, v_2) \mapsto v_1 + v_2.$$

$$\cdot : K \times V \rightarrow V, \quad (a, v) \mapsto a \cdot v$$

s.t. the following conditions (axioms) hold:

$$(V1) \quad \forall v_1, v_2, v_3 \in V : v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3.$$

$$(V2) \quad \exists \text{ an element } 0 \in V \text{ s.t. } 0 + v = v \quad \forall v \in V.$$

↑ sometimes denoted 0_V .

$$(V3) \quad \forall v \in V, \exists v' \in V \text{ s.t. } v + v' = 0.$$

$$(V4) \quad \forall v_1, v_2 \in V : v_1 + v_2 = v_2 + v_1.$$

$$(V5) \quad \forall a, b \in K, \forall v \in V : a \cdot (b \cdot v) = (a \cdot b) \cdot v.$$

$$(V6) \quad \forall v \in V : 1 \cdot v = v.$$

$$(V7) \quad \forall a \in K, v_1, v_2 \in V : a(v_1 + v_2) = a \cdot v_1 + a \cdot v_2.$$

$$(V8) \quad \forall a_1, a_2 \in K, v \in V : (a_1 + a_2) \cdot v = a_1 \cdot v + a_2 \cdot v.$$

Lemma. Let V be a vector space over a field K .

Then:

- (a) The 0 vector from $(V, +)$ is unique. To avoid confusion (e.g. with $0 \in K$) we will sometime denote it by 0_V .
- (b) The element v' (corresponding to v) from $(V, +)$ is unique. We'll denote it by $-v$.
We'll also write $w - v := w + (-v)$ for all $w \in V$.
- (c) $\forall v \in V$ we have $0 \cdot v = 0$ (i.e. $0_K \cdot v = 0_V$).
- (d) $\forall a \in K$ we have $a \cdot 0 = 0$ (i.e. $a \cdot 0_V = 0_V$).
- (e) $\forall v \in V : -1 \cdot v = -v$.
- (f) $\forall v \in V : -(-v) = v$.
- (g) If $a \cdot v = 0$ for some $a \in K, v \in V$, then either $a = 0$ or $v = 0$.
- (h) Associativity for $+$ holds also for some of n elements. $v_1 + \dots + v_n$ makes sense no matter how we put the brackets
 $v_1 + (v_2 + (\dots + (v_{n-1} + v_n) \dots))$, or
 $(v_1 + v_2) + (\dots)$ etc.

Proof of some of the points (a) - (h) from Lemma.
(The rest are exercises).

(a) Uniqueness of 0. Suppose $e \in V$ is an element that satisfy $e+v=v \forall v \in V$. We'll prove $e=0$.
 Indeed, $e+0=0$ by assumption.
 We now have $0 \stackrel{(V4)}{=} e+0 = 0+e \stackrel{(V2)}{=} e$.

(b) Suppose v' and w both satisfy $v+v'=0, v+w=0$.
 $\Rightarrow w \stackrel{(V2)}{=} 0+w = (v+v') + w \stackrel{(V4)}{=} (v'+v) + w \stackrel{(V1)}{=} v' + (v+w) = v' + 0 \stackrel{(V4)}{=} 0 + v' \stackrel{(V2)}{=} v'$.

(c) Let $v \in V$. we have to prove that $0 \cdot v = 0$.
 Indeed, $0 \cdot v = (0+0) \cdot v \stackrel{(V8)}{=} 0 \cdot v + 0 \cdot v, (*)$

Add $(-0 \cdot v)$ to both sides of the last equality and we get:

$$0 \stackrel{(V3)}{=} 0 \cdot v + (-0 \cdot v) \stackrel{(*)}{=} 0 \cdot v + \underbrace{0 \cdot v + (-0 \cdot v)}_{=0} = 0 \cdot v.$$

(g) Let $a \in K, v \in V$ and assume that $a \cdot v = 0$.
 We need to show that either $a=0$ or $v=0$.
 Indeed, if $a \neq 0$ then b.c. K is a field there is an element $a^{-1} \in K$ s.t. $a^{-1} \cdot a = 1$.

So: $v \stackrel{(V6)}{=} 1 \cdot v = (a^{-1} \cdot a) \cdot v \stackrel{(V5)}{=} a^{-1} \cdot (a \cdot v) \stackrel{(d) \uparrow \text{ of the Lemma}}{=} a^{-1} \cdot 0 = 0.$



Important example of a vector space.

The coordinate space K^n . Let K be a field

and $n \in \mathbb{N}$. $K^n = \{(a_1, \dots, a_n) \mid a_i \in K, 1 \leq i \leq n\}$.

K^n is a vector space over K if we endow it with the following operations:

Let $v = (a_1, \dots, a_n)$, $w = (b_1, \dots, b_n) \in K^n$.

$$v + w = (a_1, \dots, a_n) + (b_1, \dots, b_n) := (a_1 + b_1, \dots, a_n + b_n).$$

↑
addition
in K^n
↑
addition
in the field K

For $a \in K$, $a \cdot v = a \cdot (a_1, \dots, a_n) := (a \cdot a_1, \dots, a \cdot a_n)$.

$0 := (0, \dots, 0) \in K^n$. With these operations K^n

becomes a vect. space over K . (The proof is based

on the fact that K satisfies the axioms of a field).

(Exc.: CHECK THE DETAILS!)

We call K^n also the n -dimensional coordin. space over K .

(*) Sometimes we write the elements of K^n as row vectors and sometimes as column vectors.

Def. Let V be a vector space over K . A subset

$W \subseteq V$ is called a linear subspace of V

(or just a subspace of V), if the following holds:

$$(LSS1) \quad W \neq \emptyset.$$

$$(LSS2) \quad \forall w_1, w_2 \in W \text{ we have } w_1 + w_2 \in W.$$

$$(LSS3) \quad \forall a \in K, w \in W \text{ we have } a \cdot w \in W.$$

Lemma. Let V be a vect. space over K and $W \subseteq V$

a subspace. Then W is a vector space on its own, when endowed with the operations from V .

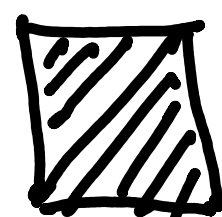
Proof. (LSS2) + (LSS3) \Rightarrow the operations $+$ and \cdot from V induce operations on W . The fact that these operations on W satisfy the 8 axioms of a vect. space follows by direct verification (\leftarrow exc!)

For example, let's check (V2) for W :

Pick any element $u \in W$ (recall $W \neq \emptyset$).

$$\text{We have } 0_V = 0_K \cdot u \in W. \quad (LSS3)$$

$$\text{Now } 0_V + w = w \quad \forall w \in W. \quad \begin{array}{c} \uparrow \\ (V2) \text{ for } V \end{array}$$



A simple criterion to check that a subset is a linear subspace

Lemma. Let V be a vect. space over K , and $W \subseteq V$ a subset. Then W is a linear subspace of V iff the following holds:

(1) $0_V \in W$.

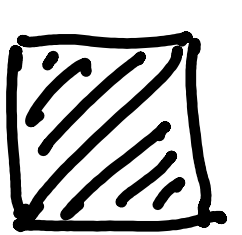
(2) $\forall a_1, a_2 \in K, w_1, w_2 \in W$ we have $a_1 w_1 + a_2 w_2 \in W$.

Proof. " $W \subseteq V$ is a subspace \implies (1) + (2)" follows immediately from the fact that W is a vector space.

"(1) + (2) \implies $W \subseteq V$ is a linear subspace"

Let $w_1, w_2 \in W$. By (2), $\underbrace{1 \cdot w_1 + 1 \cdot w_2}_{w_1 + w_2} \in W$.

Let $a \in K, w \in W$. By (2) $\underbrace{a \cdot w + 0 \cdot w}_{a \cdot w} \in W$.



Examples of subspaces.

1) Let V be a vect. space over K . Then $\{0_V\} \subseteq V$ is a linear subspace.

2) Fix $b \in K$. Consider the subset

$$U_b := \{(x_1, x_2, x_3) \in K^3 \mid x_1 + x_2 + x_3 = b\} \subseteq K^3.$$

Claim. $U_b \subseteq K^3$ is a linear subspace $\Leftrightarrow b=0$.

Proof of \Rightarrow . Suppose U_b is a subspace of K^3 .

If $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U_b$, then

$$x_1 + x_2 + x_3 = b, \quad y_1 + y_2 + y_3 = b. \quad (*)$$

Since $U_b \subseteq K^3$ a subspace we have

$$(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U_b.$$

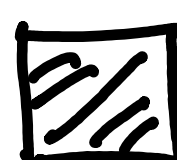
$$\Rightarrow (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = b.$$

But from (*) we also get

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = b + b. \Rightarrow b + b = b$$

$$\Rightarrow b = 0.$$

Exc.: Prove the other direction



Important generalization of the last example.

Let A be an $m \times n$ matrix with entries in K .
(from now on: $A \in M_{m \times n}(K)$, or $A \in \text{Mat}_{m \times n}(K)$).

Fix $b \in K^m$, viewed as a column vector

$$\text{i.e. } b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

HERE WE VIEW
ELEMENTS OF K^n AS
COLUMN VECTORS

Consider $L := \{x \in K^n \mid A \cdot x = b\} \subseteq K^n$.

↑
also viewed
as col. vect.

Claim. $L \subseteq K^n$ is a linear subspace $\Leftrightarrow b = 0 \in K^m$.

Important preparation for the proof of the claim.

Lemma. $\forall u, v \in K^n, a \in K$ we have

- 1) $A \cdot (u + v) = A \cdot u + A \cdot v$.
- 2) $A \cdot (a \cdot u) = a \cdot (A \cdot u)$.

Proof of Lemma. Write $A = (a_{ij})$ $1 \leq i \leq m, 1 \leq j \leq n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$A \cdot u + A \cdot v =$$

$$\begin{pmatrix} \vdots \\ a_{i1}u_1 + a_{i2}u_2 + \dots + a_{ik}u_k + \dots + a_{in}u_n \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ a_{i1}v_1 + a_{i2}v_2 + \dots + a_{ik}v_k + \dots + a_{in}v_n \\ \vdots \end{pmatrix}$$

↑ entry number i ↑ entry number i

$$= \begin{pmatrix} \vdots \\ a_{i1}(u_1+v_1) + a_{i2}(u_2+v_2) + \dots + a_{ik}(u_k+v_k) + \dots + a_{in}(u_n+v_n) \\ \vdots \end{pmatrix} =$$

$$= A \cdot \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_k+v_k \\ \vdots \\ u_n+v_n \end{pmatrix} = A \cdot (u+v).$$

Exc. Prove the 2'nd statement of the Lemma. ▣

Proof of the claim.

⇐: we assume $b=0$. we have to show $L \subseteq K^n$ is a subsp.

$L \neq \emptyset$ b.c. $x=0 \in L$ (indeed $A \cdot 0 = 0$).

If $x = (x_1, \dots, x_n) \in L$, $y = (y_1, \dots, y_n) \in L$,

$$\Rightarrow A \cdot (x+y) = A \cdot x + A \cdot y = 0 + 0 = 0. \Rightarrow x+y \in L.$$

by the
Lemma

This shows that $x, y \in L \Rightarrow x+y \in L$.

$$\text{Let } x \in L, a \in K. \Rightarrow A \cdot (a \cdot x) = a \cdot (A \cdot x) = a \cdot 0 = 0.$$

$$\text{So: } x \in L, a \in K \Rightarrow a \cdot x \in L.$$

This shows L is a subspace.

\Leftarrow : Suppose $L \subseteq K^n$ is a subspace.

We need to show that $b=0$.

Since $L \subseteq K^n$ is a subspace, $0 \in L$.

$$\Rightarrow A \cdot 0 = b \quad \text{but} \quad A \cdot 0 = 0 \Rightarrow b = 0. \quad \square$$

More examples.

Spaces of sequences. Let K be a field.

$$\text{Define } K^\infty := \{ (a_1, a_2, a_3, \dots, a_k, \dots) \mid a_i \in K \forall i \}$$

Define $+$ and \cdot on K^∞ :

$$(a_1, a_2, \dots, a_k, \dots) + (b_1, b_2, \dots, b_k, \dots) := (a_1 + b_1, a_2 + b_2, \dots, a_k + b_k, \dots)$$