Vector spaces.

Def. A vector space over a field K is a set V,
scalars vectors

endowed with two operations:

$$+ : V\times V \longrightarrow V, \qquad (v_1, v_2) \longmapsto v_1+v_2.$$

$$\cdot : K\times V \longrightarrow V, \qquad (a, v) \longmapsto a\cdot v$$

s.t. the following conditions (axioms) hold:

(V1) $\forall\, v_1, v_2, v_3 \in V : v_1+(v_2+v_3) = (v_1+v_2)+v_3$.

(V2) $\exists$ an element $0\in V$ s.t. $0+v=v \quad \forall\, v\in V$.
sometimes denoted 0_V.

(V3) $\forall\, v\in V, \exists\, v'\in V$ s.t. $v+v'=0$.

(V4) $\forall\, v_1, v_2 \in V : v_1+v_2 = v_2+v_1$.

(V5) $\forall\, a, b\in K, \forall\, v\in V : \quad a\cdot(b\cdot v) = (a\cdot b)\cdot v$.

(V6) $\forall\, v\in V : \quad 1\cdot v = v$.

(V7) $\forall\, a\in K, v_1, v_2 \in V : a(v_1+v_2) = a\cdot v_1 + a\cdot v_2$.

(V8) $\forall\, a_1, a_2 \in K, v\in V : (a_1+a_2)\cdot v = a_1\cdot v + a_2\cdot v$.

Lemma. Let V be a vector space over a field K.
Then:

(a) The 0 vector from (V2) is unique. To avoid confusion (e.g. with $0 \in K$) we will sometime denote it by 0_V.

(b) The element v' (corresponding to v) from (V3) is unique. We'll denote it by $-v$.
We'll also write $w - v := w + (-v)$ for all $w \in V$.

(c) $\forall\, v \in V$ we have $0 \cdot v = 0$ (i.e. $0_K \cdot v = 0_V$).

(d) $\forall\, a \in K$ we have $a \cdot 0 = 0$ (i.e. $a \cdot 0_V = 0_V$).

(e) $\forall\, v \in V$: $-1 \cdot v = -v$.

(f) $\forall\, v \in V$: $-(-v) = v$.

(g) If $a \cdot v = 0$ for some $a \in K$, $v \in V$, then either $a = 0$ or $v = 0$.

(h) Associativity for $+$ holds also for some of n elements. $v_1 + \dots + v_n$ makes sense no matter how we put the brackets
$v_1 + (v_2 + (\dots + (v_{n-1} + v_n) \cdots))$, or
$(v_1 + v_2) + (\dots)$ etc.

Proof of some of the points (a) - (h) from Lemma. (the rest are exercises).

(a) Uniqueness of 0. Suppose $e \in V$ is an element that satisfy $e+v=v$ $\forall v \in V$. We'll prove $e=0$.
Indeed, $e+0=0$ by assumption.
We now have $0 = e+0 \underset{(V4)}{=} 0+e \underset{(V2)}{=} e$.

(b) Suppose v' and w both satisfy $v+v'=0$, $v+w=0$.

$$\Rightarrow \quad w \underset{(V2)}{=} 0+w = (v+v')+w \underset{(V4)}{=} (v'+v)+w \underset{(V1)}{=} v'+(v+w) =$$

$$= v'+0 \underset{(V4)}{=} 0+v' \underset{(V2)}{=} v'.$$

(c) Let $v \in V$. we have to prove that $0 \cdot v = 0$.
Indeed, $0 \cdot v = (0+0)\cdot v \underset{(V8)}{=} 0\cdot v + 0 \cdot v$. (*)

Add $(-0\cdot v)$ to both sides of the last equality and we get:

$$0 \underset{(V3)}{=} 0\cdot v + (-0\cdot v) \underset{(*)}{=} 0\cdot v + \underbrace{0\cdot v + (-0\cdot v)}_{=0} = 0 \cdot v.$$

(g) Let $a \in K$, $v \in V$ and assume that $a \cdot v = 0$.
We need to show that either $a=0$ or $v=0$.
Indeed, if $a \neq 0$ then b.c. K is a field there is an element $a^{-1} \in K$ s.t. $a^{-1}\cdot a = 1$.

So: $v \underset{(V6)}{=} 1\cdot v = (a^{-1}\cdot a)\cdot v \underset{(V5)}{=} a^{-1}\cdot(a\cdot v) = a^{-1}\cdot 0 = 0$.

(d) of the lemma

$\blacksquare$

Important example of a vector space.

The coordinate space K^n. Let K be a field and $n \in \mathbb{N}$. $K^n = \{(a_1, \dots, a_n) \mid a_i \in K,\ 1 \leq i \leq n\}$.

K^n is a vector space over K if we endow it with the following operations:

Let $v = (a_1, \dots, a_n)$, $w = (b_1, \dots, b_n) \in K^n$.

$$v + w = (a_1, \dots, a_n) + (b_1, \dots, b_n) := (a_1 + b_1, \dots, a_n + b_n).$$

addition in K^n

addition in the field K

For $a \in K$, $\quad a \cdot v = a \cdot (a_1, \dots, a_n) := (a \cdot a_1, \dots, a \cdot a_n)$.

$0 := (0, \dots, 0) \in K^n$. With these operations K^n becomes a vect. space over K. (The proof is based on the fact that K satisfies the axioms of a field).

(Exc.: CHECK THE DETAILS!)

We call K^n also the **n-dimensional coordin. space over K.**

$\circledast$ Sometimes we write the elements of K^n as row vectors and sometimes as column vectors.

Def. Let V be a vector space over K. A subset $W \subseteq V$ is called a linear subspace of V (or just a subspace of V), if the following holds:

(LSS1) $W \neq \emptyset$.

(LSS2) $\forall\ w_1, w_2 \in W$ we have $w_1 + w_2 \in W$.

(LSS3) $\forall\ a \in K, w \in W$ we have $a \cdot w \in W$.

Lemma. Let V be a vect. space over K and $W \subseteq V$ a subspace. Then W is a vector space on its own, when endowed with the operations from V.

Proof. (LSS2)+(LSS3) $\Longrightarrow$ the operations $+$ and $\cdot$ from V induce operations on W. The fact that these operations on W satisfy the 8 axioms of a vect. space follows by direct verification (exc!)

For example, let's check (V2) for W:
Pick any element $u \in W$ (recall $W \neq \emptyset$).
We have $0_V = 0_K \cdot u \underset{(LSS3)}{\in} W$.

Now $0_V + w \underset{(V2) \text{ for } V}{=} w \quad \forall\ w \in W$. $\blacksquare$

A simple criterion to check that a subset is a linear subspace

__Lemma.__ Let V be a vect. space over K, and $W \subseteq V$ a subset. Then W is a linear subspace of V iff the following holds:

(1) $0_V \in W$.

(2) $\forall$ $a_1, a_2 \in K$, $w_1, w_2 \in W$ we have

$$a_1 w_1 + a_2 w_2 \in W.$$

__Proof.__ "$W \subseteq V$ is a subspace $\Rightarrow$ (1) + (2)" follows immediately from the fact that W is a vector space.

"(1) + (2) $\Rightarrow$ $W \subseteq V$ is a linear subspace"

Let $w_1, w_2 \in W$. By (2), $\underbrace{1 \cdot w_1 + 1 \cdot w_2}_{w_1 + w_2} \in W$.

Let $a \in K, w \in W$. By (2) $\underbrace{a \cdot w + 0 \cdot w}_{a \cdot w} \in W$. ▨

Examples of subspaces.

1) Let V be a vect. space over K. Then $\{0_V\} \subseteq V$ is a linear subspace.

2) Fix $b \in K$. Consider the subset

$$\mathcal{U}_b := \{(x_1, x_2, x_3) \in K^3 \mid x_1 + x_2 + x_3 = b\} \subseteq K^3.$$

Claim. $\mathcal{U}_b \subseteq K^3$ is a linear subspace $\Leftrightarrow b = 0$.

Proof of $\Rightarrow$. Suppose $\mathcal{U}_b$ is a subspace of K^3.

If $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathcal{U}_b$, then

$$x_1 + x_2 + x_3 = b, \quad y_1 + y_2 + y_3 = b. \qquad (*)$$

Since $\mathcal{U}_b \subseteq K^3$ is a subspace we have

$$(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in \mathcal{U}_b.$$

$$\Rightarrow (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = b.$$

But from $(*)$ we also get

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = b + b. \Rightarrow b + b = b$$

$$\Rightarrow b = 0.$$

Exc.: Prove the other direction $\Leftarrow$ ▨

Important generalization of the last example.

Let A be an $m\times n$ matrix with entries in K.
(from now on: $A \in M_{m\times n}(K)$, or $A \in \mathrm{Mat}_{m\times n}(K)$).
Fix $b \in K^m$, viewed as a column vector

i.e. $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

HERE WE VIEW ELEMENTS OF K^n AS COLUMN VECTORS

Consider $L := \{ x \in K^n \mid A\cdot x = b \} \subseteq K^n$.

also viewed as col. vect.

<u>Claim.</u> $L \subseteq K^n$ is a linear subspace $\Leftrightarrow$ $b = 0 \in K^m$.

Important preparation for the proof of the claim.

<u>Lemma.</u> $\forall$ $u, v \in K^n$, $a \in K$ we have

1) $A\cdot(u+v) = A\cdot u + A\cdot v$.
2) $A\cdot(a\cdot u) = a\cdot(A\cdot u)$.

<u>Proof of Lemma.</u> Write $A = (a_{ij})$ $1\le i\le m,\ 1\le j\le n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$A\cdot u + A\cdot v =$

$$\begin{array}{c} \\ \text{entry number } i \end{array}\begin{pmatrix} \vdots \\ a_{i1}u_1 + a_{i2}u_2 + \dots + a_{ik}u_k + \dots + a_{in}u_n \\ \vdots \end{pmatrix} + \begin{array}{c} \\ \text{entry number } i \end{array}\begin{pmatrix} \vdots \\ a_{i1}v_1 + a_{i2}v_2 + \dots + a_{ik}v_k + \dots + a_{in}v_n \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ a_{i1}(u_1+v_1) + a_{i2}(u_2+v_2) + \dots + a_{ik}(u_k+v_k) + \dots + a_{in}(u_n+v_n) \\ \vdots \end{pmatrix} =$$

$$= A \cdot \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_k+v_k \\ \vdots \\ u_n+v_n \end{pmatrix} = A\cdot(u+v).$$

Exc. Prove the 2'nd statement of the Lemma. ▨

Proof of the claim.

$\Leftarrow$: we assume $b=0$. we have to show $L \subseteq K^n$ is a subsp.

$L \neq \emptyset$ b.c. $x = 0 \in L$ (indeed $A\cdot 0 = 0$).

If $x = (x_1, \dots, x_n) \in L$, $y = (y_1, \dots, y_n) \in L$,

$$\Rightarrow A\cdot(x+y) \underset{\substack{\uparrow \\ \text{by the} \\ \text{Lemma}}}{=} A\cdot x + A\cdot y = 0+0 = 0. \Rightarrow x+y \in L.$$

This shows that $x, y \in L \Rightarrow x+y \in L$.

Let $x \in L$, $a \in K$. $\Rightarrow A\cdot(a\cdot x) = a\cdot(A\cdot x) = a\cdot 0 = 0$.

So: $x \in L, a \in K \Rightarrow a\cdot x \in L$.

This shows L is a subspace.

$\Leftarrow$: Suppose $L \subseteq K^n$ is a subspace.

We need to show that $b=0$.

Since $L \subseteq K^n$ is a subspace, $0 \in L$.

$\Rightarrow A\cdot 0 = b$ but $A\cdot 0 = 0 \Rightarrow b = 0$. ∎

More examples.

Spaces of sequences. Let K be a field.

Define $K^\infty := \{(a_1, a_2, a_3, \dots, a_k, \dots) \mid a_i \in K \ \forall i\}$

Define $+$ and $\cdot$ on K^∞:

$$(a_1, a_2, \dots, a_k, \dots) + (b_1, b_2, \dots, b_k, \dots) := (a_1+b_1, a_2+b_2, \dots, a_k+b_k, \dots)$$