Exercise 3 is taken from Dr. Menny Akka Ginosar's notes on Fibonnacci sequences. These notes are available on the course website.

1. The Golden Ratio Part 1. Recall that you computed $\mathcal{F}_{0,1}$ during the lectures. Now:

(a) Find a formula for $\mathcal{F}_{1,0}$.

(b) Write a formula for $\mathcal{F}_{a,b}$ that depends only on a, b and n.

Solution:

(a) There are several ways of solving this exercise. The first one is to write

$$a \cdot \mathcal{G}_\varphi + b \cdot \mathcal{G}_\psi = \mathcal{F}_{1,0}$$

and solve the corresponding linear system

$$\begin{cases} a + b = 1 \\ a\varphi + +b\psi = 0 \end{cases}$$

You will obtain

$$a = \frac{\psi}{\psi - \varphi}, \qquad b = \frac{\varphi}{\varphi - \psi}.$$

Here we kept the notation from the lectures, where $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$.

Another way to solve the exercise is to realize that $\mathcal{F}_{1,0}$ is $\mathcal{F}_{0,1}$ with a 1 in front and every index shifted by +1. This is indeed the case because each sequence in **Fib** is uniquely determined by its first two elements. Therefore, the i-th element of $\mathcal{F}_{0,1}$ is the $(i+1)$-st element of $\mathcal{F}_{1,0}$.

In any case, if we write $\mathcal{F}_{1,0} = (a_0, a_1, a_2, \dots)$, we have

$$a_n = \frac{\psi}{\psi - \varphi}\varphi^n + \frac{\varphi}{\varphi - \psi}\psi^n = \frac{\varphi\psi^n - \psi\varphi^n}{\varphi - \psi}.$$

(b) From the computations of $\mathcal{F}_{1,0} = (a_0, a_1, a_2, \dots)$ and $\mathcal{F}_{0,1} = (b_0, b_1, b_2, \dots)$, and writing $\mathcal{F}_{a,b} = (c_0, c_1, c_2, \dots)$, we have that

$$c_n = a \cdot a_n + b \cdot b_n = a \cdot \frac{\varphi\psi^n - \psi\varphi^n}{\varphi - \psi} + b \cdot \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

2. Negation and De Morgan.

(a) Let A and B be statements. Show/explain the following:

(i) A is equivalent to $\neg(\neg A)$;

(ii) $\neg(A \vee B)$ is equivalent to $\neg A \wedge \neg B$;

(iii) $\neg(A \wedge B)$ is equivalent to $\neg A \vee \neg B$.

(b) Now, let A and B be two sets contained in a bigger set U. Show/explain the following:

(i) $(A \cup B)^c = A^c \cap B^c$;

(ii) $(A \cap B)^c = A^c \cup B^c$.

Solution:

(a) (i)

A	$\neg A$	$\neg(\neg A)$
w	f	w
f	w	f

(ii)

$\neg(A \vee B)$		B: w	B: f
A	w	f	f
	f	f	w

$\neg A \wedge \neg B$		B: w	B: f
A	w	f	f
	f	f	w

(iii)

$\neg(A \wedge B)$		B: w	B: f
A	w	f	w
	f	w	w

$\neg A \vee \neg B$		B: w	B: f
A	w	f	w
	f	w	w

(b) Draw it to have a better idea of what it looks like!

(i)

$(A \cup B)^c$		B: $\in$	B: $\notin$
A	$\in$	$\notin$	$\notin$
	$\notin$	$\notin$	$\in$

$A^c \cap B^c$		B: $\in$	B: $\notin$
A	$\in$	$\notin$	$\notin$
	$\notin$	$\notin$	$\in$

Using first-order logic, we can formulate the proof as follows:

$$\begin{aligned}(A \cup B)^c &= \{x \in U \mid \neg(x \in A \vee x \in B)\} \\ &= \{x \in U \mid \neg(x \in A) \wedge \neg(x \in B)\} \\ &= \{x \in U \mid x \notin A \wedge x \notin B\} \\ &= \{x \in U \mid x \notin A\} \cap \{x \in U \mid x \notin B\} \\ &= A^c \cap B^c.\end{aligned}$$

(ii)

$(A \cap B)^c$		B: $\in$	B: $\notin$
A	$\in$	$\notin$	$\in$
	$\notin$	$\in$	$\in$

$A^c \cup B^c$		B: $\in$	B: $\notin$
A	$\in$	$\notin$	$\in$
	$\notin$	$\in$	$\in$

3. The Golden Ratio Part 2. We do not normally want to draw on your school knowledge apart from the skill of doing algebraic manipulations correctly. We make an exception in this task.

Let $\mathcal{F}_{0,1} = (F_0, F_1, F_2, \dots) \in \mathbf{Fib}$. Assuming that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists and isn't equal to 0, compute its value.

Solution: Using the recursive formula defining F_n, we have

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1}+F_{n-2}}{F_{n-1}} = \lim_{n\to\infty} \left(1 + \frac{F_{n-2}}{F_{n-1}}\right) = 1 + \frac{1}{\lim_{n\to\infty} \frac{F_n}{F_{n-1}}}.$$

Denoting the quantity we wish to compute by X, we see that it must be solution to the equation

$$X = 1 + \frac{1}{X}.$$

Since X is assumed to be different from 0, we conclude that it must be a root of the polynomial $X^2 - X - 1$, i.e. it is either φ or ψ. Since the ratios are all positive, so is their limit. Therefore

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi.$$

4. Logic. Let x, y, z be variables and let R, S be relations. Distribute the negation inside the parentheses in the following proposition:

$$\neg\Big(\forall x \exists y \exists z (R(x,y) \wedge R(z,x)) \vee (R(x,y) \wedge S(z,x))\Big)$$

Solution: The expression can be simplified to

$$\neg\Big(\forall x \exists y \exists z R(x,y) \wedge (R(z,x) \vee S(z,x))\Big),$$

which after distributing the negation will yield

$$\exists x \forall y \forall z \neg R(x,y) \vee (\neg R(z,x) \wedge \neg S(z,x)).$$

5. Logic. Let $x, y \in \mathbb{R}$. Write the contrapositive of the following:

$$x \neq y \implies (\exists \epsilon > 0 : |x-y| > \epsilon)$$

Solution: The contrapositive is

$$(\forall \epsilon > 0 : |x-y| \leq \epsilon) \implies x = y.$$

6. Dimension. Let $\mathcal{F} = (\alpha_0, \alpha_1, \alpha_2, \dots)$, $\mathcal{G} = (\beta_0, \beta_1, \beta_2, \dots) \in$ **Fib**. You are asked to check whether $\mathcal{F} = \mathcal{G}$. The sequences themselves are not known but for any natural number i, you can check whether $\alpha_i = \beta_i$. You can check this for as many i's as you wish.

Claim: It is enough to check it for two **random** i's to know with certainty whether $\mathcal{F} = \mathcal{G}$, explain why.

Solution: We indeed need at most two checks to determine whether $\mathcal{F} = \mathcal{G}$ since if for either of the checks the elements are different, the sequences are definitely different. If not, let us first write the i-th element of $\mathcal{F}_{0,1}$, respectively $\mathcal{F}_{1,0}$, as $\mathcal{F}_{0,1}^{(i)}$, respectively $\mathcal{F}_{1,0}^{(i)}$. We then know from the lecture that there exist some $a, b, a', b' \in \mathbb{R}$ such that

$$\begin{aligned} \mathcal{F} &= a \cdot \mathcal{F}_{1,0} + b \cdot \mathcal{F}_{0,1}, \\ \mathcal{G} &= a' \cdot \mathcal{F}_{1,0} + b' \cdot \mathcal{F}_{0,1}. \end{aligned}$$

We then have from our 2 checks that

$$\begin{cases} a \cdot \mathcal{F}_{1,0}^{(i)} + b \cdot \mathcal{F}_{0,1}^{(i)} = \alpha_i = \beta_i = a' \cdot \mathcal{F}_{1,0}^{(i)} + b' \cdot \mathcal{F}_{0,1}^{(i)} \\ a \cdot \mathcal{F}_{1,0}^{(j)} + b \cdot \mathcal{F}_{0,1}^{(j)} = \alpha_j = \beta_j = a' \cdot \mathcal{F}_{1,0}^{(j)} + b' \cdot \mathcal{F}_{0,1}^{(j)} \end{cases}$$

This implies that

$$\begin{cases} a = a' \\ b = b' \end{cases}$$

and therefore that $\mathcal{F} = \mathcal{F}_{a,b} = \mathcal{F}_{a',b'} = \mathcal{G}$.

Alternative solution suggested by Segev: Let $\mathcal{F} = (\alpha_0, \alpha_1, \alpha_2, \dots)$ and let $\mathcal{G} = (\beta_0, \beta_1, \beta_2, \dots)$ be the two elements of **Fib** we want to compare. Their difference $\mathcal{F} - \mathcal{G}$ is also in **Fib**. Let us denote its elements by $\gamma_0, \gamma_1, \gamma_2, \dots$ We will compare α_i to β_i and α_j to β_j for some i, j (if $j > i$, just invert the roles of i and j). If $\alpha_i = \beta_i$, we must have (everything in the following sequence is computed from γ_i, so you should read it forward and backward from there)

$$\begin{array}{rcl}
\gamma_0 & = & (-1)^{i+1} F_i \gamma_{i+1} \\
\vdots & & \\
\gamma_{i-\ell} & = & (-1)^{\ell+1} F_\ell \gamma_{i+1} \\
\vdots & & \\
\gamma_{i-3} & = & \gamma_{i-1} - \gamma_{i-2} = 2\gamma_{i+1} \\
\gamma_{i-2} & = & \gamma_i - \gamma_{i-1} = -\gamma_{i+1} \\
\gamma_{i-1} & = & \gamma_{i+1} - \gamma_i = \gamma_{i+1} \\
\gamma_i & = & 0 \\
\gamma_{i+1} & & \\
\gamma_{i+2} & = & \gamma_{i+1} \\
\gamma_{i+3} & = & 2\gamma_{i+1} \\
\gamma_{i+4} & = & 3\gamma_{i+1} \\
\vdots & & \\
\gamma_{i+k} & = & F_k \gamma_{i+1} \\
\vdots & &
\end{array}$$

where we denoted $\mathcal{F}_{0,1} = (F_0, F_1, F_2, \dots)$. If now $\alpha_j = \beta_j$, we have

$$0 = \alpha_j - \beta_j = \gamma_j = F_{j-i}\gamma_{i+1}.$$

Therefore $\gamma_{i+1} = 0$, which implies $\gamma_k = 0$ for all $k \geq 0$ by the formulae written above.