D-MATH | Lineare Algebra I | HS 2022

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Musterlösung Serie 2

Exercises 2,3, and 6 are taken from Professor Einsiedler's Analysis lecture notes.
In this exercise sheet, we assume that $\mathbb{N}$, the set of natural numbers, denotes all integers greater or equal to 1.

1. **Logic.** Formulate the statement "there does not exist a largest natural number" and the statement "for every natural number n there exists a strictly larger natural number" in First-order logic. Show by transformation in First-order logic the equivalence of both statements.

   Solution: "There does not exists a largest natural number" can be written as

   $$\neg(\exists m \in \mathbb{N} \forall n \in \mathbb{N} : m \geqslant n).$$

   The proposition "for every natural number n there exists a strictly larger natural number" can be written as

   $$\forall m \in \mathbb{N} \exists n \in \mathbb{N} : n > m.$$

   We show equivalence between the two formulae as follows:

   $$\begin{aligned} & \neg(\exists m \in \mathbb{N} \forall n \in \mathbb{N} : m \geqslant n) \\ \iff & \forall m \in \mathbb{N} \exists n \in \mathbb{N} : \neg(m \geqslant n) \\ \iff & \forall m \in \mathbb{N} \exists n \in \mathbb{N} : m < n. \end{aligned}$$

2. **Archimedean principle.** Describe the set

   $$\bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} \mid -\frac{1}{n} \leqslant x \leqslant \frac{1}{n} \right\},$$

   where n runs through the set $\mathbb{N}$ of natural numbers.

   Solution: Let $x \in \mathbb{R}$. Assume first that $x > 0$. There exists $N \in \mathbb{N}$ such that $N > 1/x$. Hence $\frac{1}{N} < x$. So,

   $$\bigcap_{n=1}^{\infty} \{x \in \mathbb{R}_{>0} \mid x \leqslant 1/n\} = \varnothing.$$

   Similarly, if $x < 0$, there exists $N \in \mathbb{N}$ such that $-N < x$ and hence $-1/N > x$. This implies

   $$\bigcap_{n=1}^{\infty} \{x \in \mathbb{R}_{<0} \mid x \geqslant -1/n\} = \varnothing.$$

   Therefore,

   $$\bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} \mid -\frac{1}{n} \leqslant x \leqslant \frac{1}{n} \right\} = \{0\}.$$

3. **Cartesian product.** Let X, Y be sets and A, A' subsets of X. Moreover, let B, B' be subsets of Y. Show

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Convince yourself, for example by drawing a picture, that there does not exist a similar formula for unions of sets.

Solution: We first prove the inclusion from left to right:

$$\begin{aligned} &(x, y) \in (A \times B) \cap (A' \times B') \\ \Longrightarrow\ &x \in A \cap A' \text{ and } y \in B \cap B' \\ \Longrightarrow\ &(x, y) \in (A \cap A') \times (B \cap B'). \end{aligned}$$

Let us now prove the other inclusion:

$$\begin{aligned} &(x, y) \in (A \cap A') \times (B \cap B') \\ \Longrightarrow\ &x \in (A \cap A') \text{ and } y \in (B \cap B') \\ \Longrightarrow\ &(x, y) \in A \times B \text{ and } (x, y) \in A' \times B' \\ \Longrightarrow\ &(x, y) \in (A \times B) \cap (A' \times B'). \end{aligned}$$

4. **Maps and sets.** Let

$$\begin{aligned} f: \ &\mathbb{R} \to \mathbb{R} \\ &x \mapsto ax + b \end{aligned}$$

for some non-zero $a \in \mathbb{R}$ and for some $b \in \mathbb{R}$. Draw the following set

$$\{(x, y) \in \mathbb{R}^2 \mid y \leqslant f(x)\} \cap \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0\} \cap \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$$

(a) for $a = 2$, $b = 1$;

(b) for $a = -2$, $b = 1$.

Solution:

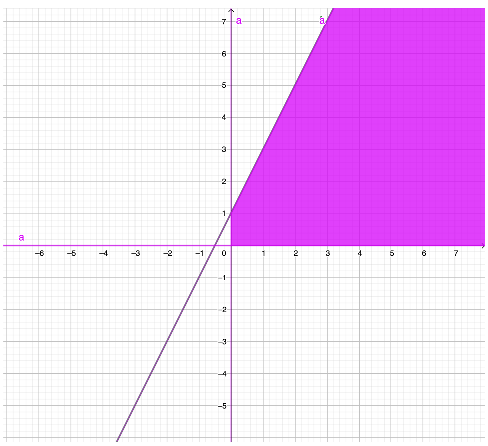

Abbildung 1: Exercise 4.a)

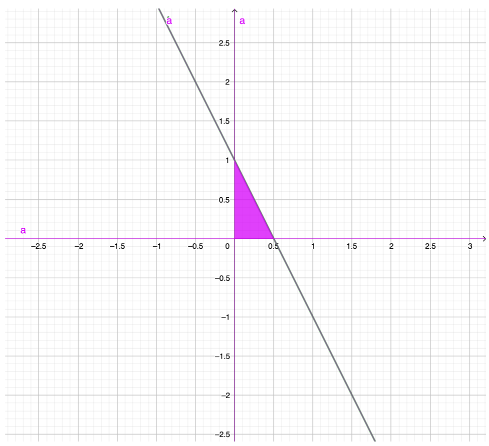

Abbildung 2: Exercise 4.b)

5. **Inverses.**

   (a) Let $f : X \to Y$ be a map. Assume that there exist maps $g_1 : Y \to X$ and $g_2 : Y \to X$ such that

   $$g_1 \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g_2 = \mathrm{id}_Y .$$

   Show then that $g_1 = g_2$. Show that then $g_1 = g_2 = f^{-1}$ and that f is bijective.
   Solution: We have

   $$g_1 = g_1 \circ \mathrm{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id}_X \circ g_2 = g_2.$$

   So $g_1 = g_2 = f^{-1}$. Moreover, since $f \circ g_2 = \mathrm{id}_Y$, f is surjective. Additionally, since $g_1 \circ f = \mathrm{id}_X$, f is injective.

   (b) Give an example of a function that admits a right-inverse but is not bijective. By right-inverse, we mean that letting f be a map from X to Y, there exists a map g from Y to X such that $f \circ g = \mathrm{id}_Y$.

   Solution: Consider the maps

   $$\begin{array}{rccc} f: & \mathbb{R} & \to & \mathbb{R}_{\geqslant 0} \\ & x & \mapsto & x^2 \end{array}$$

   and

   $$\begin{array}{rccc} g: & \mathbb{R}_{\geqslant 0} & \to & \mathbb{R} \\ & x & \mapsto & \sqrt{x} \end{array}$$

   We clearly have $f \circ g = \mathrm{id}_{R_{\geqslant 0}}$, so g is a right-inverse for f. However, f is not bijective as it is not injective.

6. **Maps and operations on sets.** Consider a function $f : X \to Y$. Let $A, A' \subseteq X$ and $B, B' \subseteq Y$ be subsets of X and Y respectively.

   (a) Prove that $f(f^{-1}(B)) \subseteq B$ is true. Under what conditions for f is equality guaranteed?

   (b) Prove that $f^{-1}(f(A)) \supseteq A$ is true. Under what conditions for f is equality guaranteed?

   (c) Show the equalities

   $$f(A \cup A') = f(A) \cup f(A'), \qquad f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B').$$

   (d) Prove that $f(A \cap A') \subseteq f(A) \cap f(A')$ and that equality is satisfied, if f is injective. Verify in that case that also $f(A \setminus A') = f(A) \setminus f(A')$ is true.

(e) Prove that $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$ and $f^{-1}(Y \smallsetminus B) = X \smallsetminus f^{-1}(B)$ are true.

In summary you should remember that forming the preimage commutes with all the set theoretic operations discussed in the lecture (including union, intersection, complement), while forming the image only satisfies this for unions or under more restrictive conditions on f.

Solution:

(a) Let $y \in f(f^{-1}(B))$. By definition,

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Hence, since there exists $x \in f^{-1}(B)$ such that $y = f(x)$, we have $y \in B$, which shows the inclusion. Equality will be guaranteed if f is surjective. Indeed, $y \in B$ and f is surjective implies that there exists some $x \in X$ such that $f(x) = y$ and that this x is in $f^{-1}(B)$. So, $y \in f(f^{-1}(B))$.

(b) By definition

$$f^{-1}(f(A)) = \{x \in X \mid f(x) \in f(A)\}.$$

So any $x \in A$ is in $f^{-1}(f(A))$ since then $f(x) \in f(A)$. If f is injective then equality will be guaranteed. Indeed, $x \in f^{-1}(f(A))$ implies that the image of x coincides with the image of an element x' of A. So, if f is injective, we must have $x = x' \in A$.

(c)
- Let $y \in f(A \cup A')$. This implies that y is the image through f of some $x \in A \cup A'$. If $x \in A$, then $y = f(x) \in f(A)$ and if $x \in A'$, $y \in f(A')$. Hence, $y \in f(A) \cup f(A')$. This shows $f(A \cup A') \subset f(A) \cup f(A')$.
  For the reverse inclusion, let $y \in f(A) \cup f(A')$. If $y \in f(A)$ then $y \in f(A \cup A')$ since $A \subseteq A \cup A'$. Similarly, if $y \in f(A')$ then $y \in f(A \cup A')$. This shows $y \in f(A) \cup f(A') \implies f(A \cup A')$.
- We first show the left to right inclusion. Let $x \in f^{-1}(B \cup B')$. By definition, this implies $y := f(x) \in B \cup B'$. If $y \in B$ then $x \in f^{-1}(B)$ and if $y \in B'$, then $x \in f^{-1}(B')$. Hence $x \in f^{-1}(B) \cup f^{-1}(B')$.
  To show the right to left inclusion, let $x \in f^{-1}(B) \cup f^{-1}(B')$. If $x \in f^{-1}(B)$, then $y := f(x) \in B$. If $x \in f^{-1}(B')$, then $y \in B'$. Hence $y \in B \cup B'$, which implies that $x \in f^{-1}(B \cup B')$.

(d) Let $y \in f(A \cap A')$. Then there exists some $x \in A \cap A'$ such that $y = f(x)$. Since $x \in A \cap A' \subseteq A$, we have $y \in f(A)$. Similarly, $y \in f(A')$. Hence $y \in f(A) \cap f(A')$. Assume now that f is injective and let $y \in f(A) \cap f(A')$. Then, there exist $x \in A$ and $x' \in A'$ such that $f(x) = y = f(x')$. Since f is injective, we must have $x = x' \in A \cap A'$ and therefore $y \in f(A \cap A')$. For the second part of the question, notice that $A \smallsetminus A' = A \cap (A')^c$. We therefore have $f(A \smallsetminus A') \subseteq f(A) \cap f((A')^c)$. We are done if we show that

$f(A) \cap f((A')^c) = f(A) \cap f(A')^c$ under the assumption that f is injective. If $y \in f(A) \cap f((A')^c)$, then the unique preimage of y is some $x \in A \cap (A')^c$. Hence $y \notin f(A')$ and $y \in f(A)$, i.e. $y \in f(A) \cap f(A')^c$. Now, if $y \in f(A) \cap f(A')^c$ its unique preimage is in A and it cannot be in A'. Therefore $y \in f(A) \cap f((A')^c)$.

(e) Let $x \in f^{-1}(B \cap B')$. Then $y := f(x) \in B \cap B'$. So, $x \in f^{-1}(B)$ and $x \in f^{-1}(B')$. For the reverse inclusion, assume that $x \in f^{-1}(B) \cap f^{-1}(B')$. Then the image of x is in B and in B', i.e. $x \in f^{-1}(B \cap B')$.

Let $x \in f^{-1}(Y \setminus B)$. Then the image $y = f(x)$ of x is in $Y \setminus B$, which implies that x is not in the preimage of B, i.e. $x \in X \setminus f^{-1}(B)$. Conversely, let $x \in X \setminus f^{-1}(B)$. Then the image of x will land anywhere in Y apart from B. Therefore $x \in f^{-1}(Y \setminus B)$.

Alternative solution: We have

$$\begin{aligned} f^{-1}(B \cap B') &= \{x \in X \mid f(x) \in B \cap B'\} \\ &= \{x \in X \mid f(x) \in B \wedge f(x) \in B'\} \\ &= \{x \in X \mid f(x) \in B\} \cap \{x \in X \mid f(x) \in B'\} \\ &= f^{-1}(B) \cap f^{-1}(B'). \end{aligned}$$

Additionally,

$$\begin{aligned} f^{-1}(Y \setminus B) &= \{x \in X \mid f(x) \in (Y \setminus B)\} \\ &= \{x \in X \mid f(x) \in Y \wedge f(x) \notin B\} \\ &= \{x \in X \mid f(x) \in Y\} \cap \{x \in X \mid f(x) \notin B\} \\ &= X \cap \{x \in X \mid f(x) \notin B\}. \end{aligned}$$

On the other hand, since $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

$$\begin{aligned} X \setminus f^{-1}(B) &= \{x \in X \mid x \notin f^{-1}(B)\} \\ &= \{x \in X \mid f(x) \notin B\}. \end{aligned}$$

This equals the last term of the previous string of equality and hence proves the statement.