D-MATH | Lineare Algebra I | HS 2022

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Musterlösung Serie 4

1. Let $m \in \mathbb{R}$. Describe the solutions of the following system of equations depending on m:

$$\begin{cases} x + my &= -3 \\ mx + 4y &= 6 \end{cases}$$

When is the set of solutions S a linear subspace of $\mathbb{R}^2$? Give a geometrical interpretation of S depending on m.

Solution: We use Gauss elimination. We do not change the first row but we replace the second row R_2 by $R_2 - mR_1$. We obtain the equivalent system

$$\begin{cases} x + my &= -3 \\ (4-m^2)y &= 6+3m = 3(m+2) \end{cases}$$

We now discuss the solutions depending on the value of m:

- If $m \notin \{\pm 2\}$, then $4 - m^2$ does not vanish. Hence,

$$y = \frac{3(m+2)}{4-m^2} = \frac{3(m+2)}{(2+m)(2-m)} = \frac{3}{2-m}$$

Plugging it into the first equation, we find

$$x = \frac{6}{m-2}.$$

The system therefore admits a unique solution, namely $(\frac{-6}{2-m}, \frac{3}{2-m})$. Geometrically, this means that the two lines defined by the equations $x + my = -3$ and $mx + 4y = 6$ intersect in this point.

- If now $m = 2$, the last line becomes $0 = 12$. Therefore the system doesn't admit any solution. Geometrically, this implies that the two lines are parallel when $m = 2$.

- If $m = -2$, then the last line becomes $0 = 0$ and the system is equivalent to $x = -3 - my$. We then have

$$S = \{(-3 - my, y) \mid y \in \mathbb{R}\}.$$

Geometrically, this implies that both equations define the same line when $m = -2$.

In none of these cases is $S \subseteq \mathbb{R}^2$ a linear subspace since it never contains $0 \cdot S = \{(0,0)\}$.

2. Which of the following sets are linear subspaces of the given vector spaces? What changes when $\mathbb{R}$ is replaced by $\mathbb{F}_2$ in (b) and (c)?

   (a) $S_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 2x_3\} \subseteq \mathbb{R}^3$
   (b) $S_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^4 = 0\} \subseteq \mathbb{R}^2$
   (c) $S_3 := \{(\mu + \lambda, \lambda^2) \in \mathbb{R}^2 \mid \mu, \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^2$

   Solution:

   (a) We see that S_1 is the set of solutions of a homogeneous linear system of equations and thus a linear subspace.
   (b) The equation $x_1^2 + x_2^4 = 0$ has only the solution $x_1 = x_2 = 0$ in $\mathbb{R}$. Therefore, we have $S_2 = \{(0, 0)\}$ and thus it is a linear subspace.
   If we consider the equation over $\mathbb{F}_2$, the situation is different. For every $\lambda \in \mathbb{F}_2$, we have
   $$\lambda^2 = \lambda.$$
   Hence, the equation defining S_2 is equivalent to $x_1 + x_2 = 0$ over $\mathbb{F}_2$. As the set of solutions of a homogenoues linear equation S_2 is a linear subspace of $\mathbb{F}_2^2$.
   (c) The set S_3 is not a linear subspace of $\mathbb{R}^2$, as for example $(1, 1)$ is contained in it, but not the multiple
   $$(-1) \cdot (1, 1) = (-1, -1),$$
   as the square of any real number is positive.
   Over $\mathbb{F}_2$, we again use the identity $\lambda^2 = \lambda$. For any $x, y \in \mathbb{F}_2$, we define
   $$\lambda = y,\ \mu = x - y.$$
   Then $(\mu + \lambda, \lambda) = (x, y)$, hence $S_3 = \mathbb{F}_2^2$. In particular, it is also a linear subspace.

3. Let K be a field in which $1 + 1 \neq 0$ and consider the space
   $$V = K^K = \mathrm{Abb}(K, K) := \{f : K \to K\}.$$
   Recall from the lectures that it is a vector space when endowed with scalar multiplication, namely $(\alpha \cdot f)(x) = \alpha f(x)$, $\forall \alpha \in K$, $\forall x \in K$, and with point wise addition, i.e. $(f + g)(x) = f(x) + g(x)$, $\forall x \in K$.

   Now let
   $$V_{even} := \{f : K \to K \mid f(-x) = f(x)\, \forall x \in K\},$$
   $$V_{odd} := \{f : K \to K \mid f(-x) = -f(x)\, \forall x \in K\}.$$

Show that V_{even} and V_{odd} are linear subspaces of V, that

$$V_{even} + V_{odd} := \{v + w \mid v \in V_{even},\, w \in V_{odd}\} = V$$

and that $V_{even} \cap V_{odd} = \{0\}$.

Solution: First note that the function that vanishes everywhere belongs both subsets, hence they are not empty. Now let $f, g \in V_{even}$ and $a \in K$. We have

$$\forall x \in K : (f + a \cdot g)(-x) = f(-x) + a \cdot g(-x) = f(x) + a \cdot g(x) = (f + a \cdot g)(x).$$

Hence $f + a \cdot g \in V_{even}$ for all $f, g \in V_{even}$, for all $a \in K$. This proves that V_{even} is a linear subspace of V.

Similarly, let $f, g \in V_{odd}$ and let $a \in K$. Then,

$$(f + a \cdot g)(-x) = -f(x) - a \cdot g(x) = -(f + a \cdot g)(x).$$

Hence V_{odd} is a linear subspace of V.

Assume now that $f \in V_{odd} \cap V_{even}$. Then, for all $x \in K$

$$-f(x) = f(-x) = f(x),$$

which implies that we must have $f(x) = 0$ for all $x \in K$ since $1 + 1 \neq 0$.

Finally, we show that $V_{even} + V_{odd} = V$. Let $f \in V$ and define

$$f_{even}(x) := \frac{f(x) + f(-x)}{2}$$
$$f_{odd}(x) := \frac{f(x) - f(-x)}{2}.$$

You can easily see that $f_{even} \in V_{even}$, that $f_{odd} \in V_{odd}$ and that

$$f(x) = f_{even}(x) + f_{odd}(x).$$

This concludes the proof.

4. Let ∞ and $-\infty$ denote 2 distinct objects, neither of which is in $\mathbb{R}$, Define an addition and a scalar multiplication on $V := \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as follows: in $\mathbb{R}$, addition and multiplication are defined as usual. For $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty.$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = (-\infty), \quad \infty + (-\infty) = (-\infty) + \infty = 0.$$

Is V a vector space over $\mathbb{R}$?

Solutions: We have $(1-2) \cdot \infty = -1 \cdot \infty = -\infty$ but $(1 \cdot \infty) + (-2 \cdot \infty) = \infty + (-\infty) = 0 \neq -\infty$. So, axiom (V8) doesn't hold and V isn't a vector space over $\mathbb{R}$.

5. Let X be a set and let P be its power set (this means that P is the set of all subsets of X). For all $A, B \in P$ and for $\lambda \in \mathbb{F}_2$, define

$$A \triangle B := (A \cup B) \setminus (A \cap B)$$
$$\lambda \cdot A := \begin{cases} \varnothing, & \text{for } \lambda = 0, \\ A, & \text{for } \lambda = 1. \end{cases}$$

Show that $(P, \triangle, \cdot, \varnothing)$ is a $\mathbb{F}_2$-vector space.

Solution: Throughout, let $A, B, C \in P$ and $s, t \in \mathbb{F}_2$. We carefully check the axioms.

(V1) We can get intuition for associativity using Venn diagrams. We obtain the following picture:

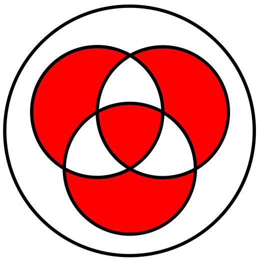

Abbildung 1: Symmetric difference of 3 sets

Let us now prove it. Let $x \in A \triangle (B \triangle C)$. We then have

$$\begin{aligned} & x \in A \triangle (B \triangle C) \\ \Leftrightarrow\ & (x \in A \wedge x \notin B \triangle C) \vee (x \notin A \wedge x \in B \triangle C) \\ \Leftrightarrow\ & \big[x \in A \wedge \neg\big((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)\big)\big] \\ & \vee \big[x \notin A \wedge \big((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)\big)\big] \end{aligned}$$

We split the last expression above into the two parts delimited by square brackets and simplify them individually. The second one can be written as

$$(x \notin A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C)$$

by distributing the logical "and " inside the parentheses.

On the other hand, for the first part

$$\begin{aligned}
& x \in A \wedge \neg\big((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)\big) \\
\Leftrightarrow\quad & x \in A \wedge \big(\neg(x \in B \wedge x \notin C) \wedge \neg(x \notin B \wedge x \in C)\big) \\
\Leftrightarrow\quad & x \in A \wedge \big((x \notin B \vee x \in C) \wedge (x \in B \vee x \notin C)\big) \\
\Leftrightarrow\quad & x \in A \wedge \big((x \notin B \wedge x \in B) \vee (x \notin B \wedge x \notin C) \\
& \vee (x \in C \wedge x \in B) \vee (x \in C \wedge x \notin C)\big) \\
\Leftrightarrow\quad & x \in A \wedge \big((x \notin B \wedge x \notin C) \vee (x \in C \wedge x \in B)\big) \\
\Leftrightarrow\quad & (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in A \wedge x \in C \wedge x \in B)
\end{aligned}$$

From line 3 to line 4 as well as from line 6 to line 7 we used the distributive property of the logical "and " with respect to the logical "or".
Putting the two parts back together, we have

$$\begin{aligned}
& x \in A \triangle (B \triangle C) \\
\Leftrightarrow\quad & (x \notin A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C) \\
& \vee (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in A \wedge x \in C \wedge x \in B)
\end{aligned}$$

This last line the the translation into logic from the above Venn diagram.
By following the exact same steps for $(A \triangle B) \triangle C$, we obtain exactly the same final expression and conclude that

$$A \triangle (B \triangle C) = (A \triangle B) \triangle C.$$

(V2) We have

$$\varnothing + A = (A \cup \varnothing) \setminus (A \cap \varnothing) = A.$$

(V3) We see that

$$A + A = (A \cup A) \setminus (A \cap A) = A \setminus A = \varnothing.$$

(V4) We see that

$$A + B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B + A.$$

(V5) We have

$$\begin{aligned}
(0 \cdot 0) \cdot A &= 0 \cdot A = \varnothing = 0 \cdot \varnothing = 0 \cdot (0 \cdot A) \\
(0 \cdot 1) \cdot A &= 0 \cdot A = \varnothing = 0 \cdot (1 \cdot A) \\
(1 \cdot 1) \cdot A &= 1 \cdot A = A = 1 \cdot (1 \cdot A).
\end{aligned}$$

The rest of the cases are proved by the commutativity of $\mathbb{F}_2$.

(V6) It holds by definition.

(V7) We have

$$0 \cdot (A+B) = \varnothing = \varnothing + \varnothing = 0 \cdot A + 0 \cdot B$$
$$1 \cdot (A+B) = A+B = (1 \cdot A) + (1 \cdot B).$$

(V8) We have

$$(0+0) \cdot A = 0 \cdot A = \varnothing = \varnothing + \varnothing = 0 \cdot A + 0 \cdot A$$
$$(1+0) \cdot A = 1 \cdot A = A = A + \varnothing = 1 \cdot A + 0 \cdot \varnothing$$
$$(1+1) \cdot A = 0 \cdot A = \varnothing = A + A = 1 \cdot A + 1 \cdot A.$$

6. Let V be a K-vector space and let $V_1, V_2.V_3 \subseteq V$ be linear subspaces, none of which is contained in another. Determine with proof if $V_1 \cup V_2 \cup V_3$ is always, sometimes or never a linear subspace of V.

   Hint: Try different fields K to obtain examples.

   Solution: Consider K^2 and the subspaces

$$V_1 := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, V_2 := \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, V_3 := \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle.$$

   When $K = \mathbb{R}$, we have $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V_1$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V_3$. However, their sum $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is not in $V_1 \cup V_2 \cup V_3$. This yields an example where the union is not a linear subspace.

   Now take $K = \mathbb{F}_2$. Then

$$V_1 \cup V_2 \cup V_3 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

   This equals the whole of K^2 and therefore it is a linear subspace.

Multiple Choice questions. Each question can admit several answers.

Question 1. Which of the following sets are linear subspaces of the given vector spaces?

- ✓ $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 + 5x_2 + 3x_3 = 0, 2x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$
  You saw in the lectures that sets of solutions of systems of homogeneous linear equations are linear subspaces.

- ○ $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3\} \subseteq \mathbb{R}^3$
  It is not a linear subspace since $(0, 0, 0)$ is not in it.

- ○ $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\} \subseteq \mathbb{R}^2$
  It is not a linear subspace since multiplying any element by a negative scalar λ reverses the inequality. Hence for any (x_1, x_2) in the set, $\lambda(x_1, x_2)$ is not in the set anymore.

- ✓ $\{(0, x, 2x, 3x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^4$
  It is a linear subspace since for any $v = (0, x, 2x, 3x)$, $w = (0, y, 2y, 3y)$ in the set and any $\lambda \in \mathbb{R}$, we have

  $$v + \lambda w = (0, x + \lambda y, 2x + 2\lambda y, 3x + 3\lambda y) = (0, z, 2z, 3z),$$

  for $z = x + \lambda y$. Hence $v + \lambda w$ stays in the set.

- ○ $\{(x^4, x^3, x^2, x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^4$
  It is not a linear subspace. Note that $v := (1, 1, 1, 1)$ lies in this set but that $2v$ does not since there is no $x \in \mathbb{R}$ such that $2 = x^4 = x^3 = x^2 = x$.

Question 2. Consider the set of pairs of positive real numbers $\mathbb{R}^2_+$. The addition on $\mathbb{R}^2_+$ is defined as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}.$$

We now consider three different definitions of scalar multiplication, for a $\lambda \in \mathbb{R}$:

- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$
- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} e^\lambda x_1 \\ e^\lambda x_2 \end{pmatrix}$
- $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1^\lambda \\ x_2^\lambda \end{pmatrix}$

According to which definition of scalar multiplication does $\mathbb{R}^2_+$ with the addition defined above become a $\mathbb{R}$-vector space?

- ○ First definition
  No. If $\lambda < 0$ then $\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \notin \mathbb{R}^2_+$

- ○ Second definition
  No. Distributivity $\lambda(v + w) = \lambda v + \lambda w$ is not verified.

- ✓ Third definition
  It checks all of the axioms.