Musterlösung Serie 5

1. **Polynomials.** Consider the polynomials

$$\begin{aligned} p_1(x) &= x^3 + x^2 \\ p_2(x) &= x^2 - 2x - 4 \\ p_3(x) &= 3x + 4 \\ p_4(x) &= 2x + 3 \end{aligned}$$

(a) Express the polynomial $2x^3 + 3x^2 - 1$ as a linear combination of the p_i, $i = 1, 2, 3, 4$.

(b) Calculate the linear span $\mathrm{Sp}(p_1, p_2, p_3, p_4)$.

Solution:

(a) We have

$$2x^3 + 3x^2 - 1 = 2p_1(x) + p_2(x) + p_4(x).$$

(b) We first note that $-2p_3(x) + 3p_4(x) = 1$. Moreover, $3p_3(x) - 4p_4(x) = x$. It follows that

$$x^2 = p_2(x) + 2\,(3p_3(x) - 4p_4(x)) + 4\,(-2p_3(x) + 3p_4(x))$$

and finally that

$$x^3 = p_1(x) - p_2(x) - 2\,(3p_3(x) - 4p_4(x)) - 4\,(-2p_3(x) + 3p_4(x))$$

This shows that

$$\left\{1, x, x^2, x^3\right\} \subseteq \mathrm{Sp}(p_1(x), p_2(x), p_3(x), p_4(x)).$$

So, by minimality of the span and since the right-hand side is linear,

$$\mathrm{Sp}(1, x, x^2, x^3) \subseteq \mathrm{Sp}(p_1(x), p_2(x), p_3(x), p_4(x)).$$

On the other hand, since $\deg(p) \leqslant 3$ for all $p \in \mathrm{Sp}(p_1(x), p_2(x), p_3(x), p_4(x))$ (see the solution to exercise 5 for more details), we have

$$\begin{aligned} &\mathrm{Sp}(p_1(x), p_2(x), p_3(x), p_4(x)) \subseteq P_3(\mathbb{R}) := \{p \in \mathbb{R}[x] \mid \deg(p) \leqslant 3\} \\ &= \mathrm{Sp}(1, x, x^2, x^3). \end{aligned}$$

Therefore,

$$\mathrm{Sp}(p_1(x), p_2(x), p_3(x), p_4(x)) = P_3(\mathbb{R}).$$

2. **Dimension 2.** Let $v \in \mathbb{R}^2 \setminus \{(0,0)\}$ and let $w \in \mathbb{R}^2 \setminus \{(0,0)\}$ be such that $w \neq \alpha v$ for all $\alpha \in \mathbb{R}$.

   (a) Show that $\mathrm{Sp}(v, w) = \mathbb{R}^2$.

   (b) Show that the only subspaces of $\mathbb{R}^2$ are $\{(0,0)\}$, $\mathrm{Sp}(v)$ for all $v \in \mathbb{R}^2$, and $\mathbb{R}^2$.

   Solution:

   (a) We write

$$v = \begin{pmatrix} a \\ b \end{pmatrix}, \quad w = \begin{pmatrix} c \\ d \end{pmatrix}.$$

We want to show that for any $(\alpha, \beta) \in \mathbb{R}^2$, there exists $(x, y) \in \mathbb{R}^2$ so that

$$xv + yw = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This amounts to showing that the linear system given by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

always admit a solution (x, y). We show it using Gauss reduction.
Without loss of generality, we can assume that $a \neq 0$. Indeed, if $a = 0$, simply switch the first row R_1 with the second row R_2 in the augmented matrix

$$\left(\begin{array}{cc|c} a & c & \alpha \\ b & d & \beta \end{array}\right).$$

Then, since w is not proportional to v, we know that $b \neq 0$ and therefore we have obtained an augmented matrix whose upper right entry is non-zero for an equivalent system of equation.
So, assuming $a \neq 0$, replace R_2 with $R_2 - \frac{b}{a} R_1$. We obtain

$$\left(\begin{array}{cc|c} a & c & \alpha \\ 0 & d - \frac{b}{a}c & \beta - \frac{b}{a}\alpha \end{array}\right).$$

Note that $d - \frac{b}{a}c \neq 0$, otherwise $w = \frac{c}{a}v$. In order to simplify the notation, write

$$c' := \frac{c}{a}, \quad \alpha' := \frac{\alpha}{a}, \quad \beta' := \frac{\beta - \frac{b}{a}\alpha}{d - \frac{b}{a}c}.$$

We have obtained an equivalent system corresponding to the augmented matrix

$$\left(\begin{array}{cc|c} 1 & c' & \alpha' \\ 0 & 1 & \beta' \end{array}\right).$$

Solving it, we conclude that a solution indeed exists. It is given by $x = \alpha' - c'\beta'$, $y = \beta'$.

(b) The three types of subsets listed above are clearly subspaces. Assume that there exists some subspace $U \subseteq \mathbb{R}^2$ that is not of any of these forms. Then $U \neq \{(0,0)\}$ and U contains at least 2 non-zero vectors that are not proportional. But then $U = \mathbb{R}^2$ by (a), which is a contradiction.

3. **Subvectorspaces.** Do the operations $+$ and $\cap$ on subvectorspaces satisfy distributivity? In other words, do the following equations hold for all linear subspaces?

$$U \cap (V_1 + V_2) = (U \cap V_1) + (U \cap V_2)$$
$$U + (V_1 \cap V_2) = (U + V_1) \cap (U + V_2)$$

If not, is at least one inclusion satisfied?

Recall that for two subspaces V_1 and V_2 of a vector space V,

$$V_1 + V_2 = \{u + v \mid u \in V_1,\, v \in V_2\}.$$

Solution: For each $i = 1, 2$, we have $V_i \subset V_1 + V_2$ and thus $U \cap V_i \subset U \cap (V_1 + V_2)$. As $U \cap (V_1 + V_2)$ is a subvectorspace, we get

$$(U \cap V_1) + (U \cap V_2) \subset U \cap (V_1 + V_2)\,.$$

Moreover, we have $V_1 \cap V_2 \subset V_i$ and thus $U + (V_1 \cap V_2) \subset U + V_i$, which implies

$$U + (V_1 \cap V_2) \subset (U + V_1) \cap (U + V_2)\,.$$

However, the other inclusion is false in both cases: For the subvectorspaces $U := \langle(1,1)\rangle$ and $V_1 := \langle(1,0)\rangle$ and $V_2 := \langle(0,1)\rangle$ of K^2 we have $U \cap V_1 = U \cap V_2 = V_1 \cap V_2 = 0$ and $U + V_1 = U + V_2 = V_1 + V_2 = K^2$ and hence

$$(U \cap V_1) + (U \cap V_2) = 0 + 0 = 0 \neq U = U \cap K^2 = U \cap (V_1 + V_2)$$

and

$$U + (V_1 \cap V_2) = U + 0 = U \neq K^2 = K^2 \cap K^2 = (U + V_1) \cap (U + V_2)$$

4. **Subspaces and equations.** Let K be a field. Fix $x \in K^n$ and $b \in K^m$. Define

$$U := \{A \in M_{m\times n}(K) \mid A \cdot x = b\}.$$

For which values of x and b is $U \subseteq M_{m\times n}(K)$ a linear subspace?

Solution: Let us denote U by $U_{x,b}$ to emphasize the dependence on the parameters. We first assume for a contradiction that $b \neq 0$ and that $U_{x,b}$ is a subspace of $M_{m\times n}(K)$. Then $0 \notin U_{x,b}$, which is a contradiction to (UVR3). Therefore $b = 0$ is a necessary condition for $U_{x,b}$ to be a subvectorspace. So, we fix $b = 0$ and denote $U_{0,x} = U_x$. Such a system of equations is called homogeneous.

We first check that for any fixed $x \in K^n$, $U_x \neq \varnothing$. That is clear since the matrix whose entries are all 0 belongs to U_x. This checks (UVR1) for U_x. Additionally, letting $A, B \in U_x$, we have

$$(A + B) \cdot x = A \cdot x + B \cdot x = 0.$$

Hence U_x verifies (UVR2). Finally, for any $\alpha \in \mathbb{R}$ and $A \in U_x$, we have

$$(\alpha A) \cdot x = \alpha(A \cdot x) = 0.$$

Hence U_x verifies (UVR3).

This proves that $b = 0$ is a necessary and sufficient condition for $U_{x,b}$ to be a linear subspace.

5. **Polynomials.** Prove that $K[x]$ is *not* finite-dimensional over K.

   Solution: We proceed by contradiction. Assume that it is finite-dimensional over K. Then there exists a finite set of polynomials

$$E := \{p_1, p_2, p_3, \ldots, p_r\} \subseteq K[x]$$

   such that $\mathrm{Sp}(E) = K[x]$, namely, for any $q \in K[x]$ there exist $a_1, \ldots, a_r \in K$ such that

$$q(x) = \sum_{i=1}^{r} a_i p_i(x).$$

   Let $D = \max_{1 \leqslant i \leqslant r}\{\deg(p_i)\}$. Since for any $p, p' \in K[x]$

$$\deg(p + p') \leqslant \max\{\deg(p), \deg(p')\},$$

   we can show by induction that for any $n \in \mathbb{N}_{\geqslant 1}$ and for any $m_1, \ldots, m_n \in K[x]$

$$\deg(m_1 + m_2 + \cdots + m_n) \leqslant \max_{1 \leqslant i \leqslant n}\{\deg(m_i)\}.$$

   Hence, for any $a_1, \ldots, a_r \in K$

$$\deg\left(\sum_{i=1}^{r} a_i p_i(x)\right) \leqslant D.$$

   Taking $q \in K[x]$ with $\deg(q) > D$ such that $q(x) = \sum_{i=1}^{r} a_i p_i(x)$, we obtain a contradiction since then

$$\deg(q) = \deg\left(\sum_{i=1}^{r} a_i p_i(x)\right) \leqslant D < \deg(q).$$

6. **Sequences.**

(a) Let K_0^∞ be the set of finitely-supported sequences, i.e.

$$K_0^\infty = \{(a_0, a_1, a_2, \dots, a_n, \dots) \mid \forall i \in \mathbb{N}\, a_i \in K \wedge \exists N \geqslant 0 : \forall n \geqslant N\, a_n = 0\}.$$

Write the smallest (with respect to inclusion) generating subset $E \subsetneq K_0^\infty$ that you can think of and justify your answer.

(b) Do the same for

$$K_{cst}^\infty := \{(a_0, a_1, a_2, \dots, a_n, \dots) \mid \forall i \in \mathbb{N}\, a_i \in K \wedge \exists c \in K \exists N \geqslant 0 : \forall n \geqslant N\, a_n = c\},$$

the set of eventually constant sequences.

Solution:

(a) An example of such a minimal generating subset is the set

$$\{a^{(i)} = (a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}, \dots) \mid 0 \leqslant i\},$$

where

$$a_j^{(i)} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

This is clearly a subset of K_0^∞. We check that it is a generating subset. Let the sequence

$$a = (a_0, a_1, a_2, \dots) \in K_0^\infty$$

and let $N \geqslant 0$ be the index such that $a_n = 0\, \forall n \geqslant N$. Then

$$a = a_0 a^{(0)} + a_1 a^{(1)} + \cdots + a_{N-1} a^{(N-1)}.$$

This shows that $\{a^{(i)} \mid i \geqslant 0\}$ is indeed a generating set.
Now assume that $B \subsetneq \{a^{(i)} \mid i \geqslant 0\}$. Then there exists $r \geqslant 0$ such that $a^{(r)} \notin B$. So, the element with index r vanishes for any linear combination of elements of B. Hence $a^{(r)} \in K_0^\infty \smallsetminus \mathrm{Sp}(B)$, which means that B is not a generating set for any $B \subsetneq \{a^{(i)} \mid i \geqslant 0\}$.

(b) An example in this case is

$$\{b^{(i)} = (b_0^{(i)}, b_1^{(i)}, b_2^{(i)}, \dots, b_n^{(i)}, \dots) \mid 0 \leqslant i\},$$

where

$$b_j^{(i)} = \begin{cases} 1, & \text{if } i \leqslant j \\ 0, & \text{otherwise} \end{cases}$$

Another such set could be the set described in (a) to which we add the element

$$(1, 1, 1, \cdots, 1, \cdots).$$

We show that $\{b^{(i)} \mid i \geqslant 0\}$ is a minimal generating set. Let

$$a = (a_0, a_1, a_2, \cdots) \in K_{cst}^{\infty}$$

and let $N \geqslant 0$ be the index such that $a_n = c \in K$, $\forall n \geqslant N$. We define

$$c_i := \begin{cases} a_0, & i = 0 \\ a_i - a_{i-1}, & 1 \leqslant i \leqslant N \end{cases}$$

Then,

$$\sum_{0 \leqslant i \leqslant N} c_i b^{(i)} = b.$$

Now assume that $B \subsetneq \{b^{(i)} \mid i \geqslant 0\}$. Then $b^{(r)} \notin B$ for some $r \geqslant 0$. Now, letting $N \in \mathbb{N}$ arbitrary and $\{\alpha_i \mid 0 \leqslant i \leqslant N\} \subset K$ with $\alpha_N \neq 0$ (otherwise we remove any vanishing element whose index is bigger so that the α_i with the greatest index is non-zero), set

$$d = {\sum_{i=0}^{N}}' \alpha_i b^{(i)},$$

where the primed sum indicates that we omit the index r in the summation whenever $N \geqslant r$. The constant tail of d is equal to ${\sum'}_{i=0}^{N} \alpha_i$ and starts at index N. Hence we cannot obtain $b^{(r)}$ with any such combination since its constant tail starts at index r. This shows that

$$\mathrm{Sp}(B) \subsetneq K_{cst}^{\infty}$$

for any $B \subsetneq \{b^{(i)} \mid i \geqslant 0\}$.