Prof. M. Einsiedler
Prof. P. Biran

Musterlösung Serie 6

1. For each of the following sets, prove whether or not they are linearly independent over $\mathbb{R}$:

$$S_1 = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-3\\9\end{pmatrix}, \begin{pmatrix}1\\3\\-4\\7\end{pmatrix}, \begin{pmatrix}2\\0\\1\\3\end{pmatrix} \right\},$$

$$S_2 = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \right\}.$$

Solution: The vectors $v_1, \dots v_4 \in \mathbb{R}^4$ are linearly independent if and only if the homogeneous linear system of equations $x_1v_1 + \cdots + x_4v_4$ is only solved by $x_1 = x_2 = x_3 = x_4 = 0$. Gaussian elimination leads to

$$\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \begin{pmatrix}2\\3\\-3\\9\end{pmatrix} = \begin{pmatrix}1\\3\\-4\\7\end{pmatrix} + \begin{pmatrix}2\\0\\1\\3\end{pmatrix}.$$

Thus, the vectors in S_1 are linearly dependent.

For S_2, we can write

$$a\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + d\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

as

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&-1&0\\1&0&0&-1\end{pmatrix} \cdot \begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Using Gauss elimination, we find that the only solution to this equation is $a = b = c = d = 0$.

2. Are the following sets linearly independent over $\mathbb{R}$?

   (a) $\{(1,0,0), (0,2,t), (2,4,t^2)\}$ for t in $\mathbb{R}$;

(b) The set of columns of an upper triangular matrix $A \in M_{n\times n}(\mathbb{R})$ with $A_{ii} \neq 0$ for all $1 \leqslant i \leqslant n$. We define an upper triangular matrix to be a matrix whose entries under the diagonal all vanish, i.e. a matrix $A = (A_{ij})_{1\leqslant i,j\leqslant n}$ such that $A_{ij} = 0$ whenever $j < i$.

(c) $\{f, g\} \subseteq \text{Abb}(\mathbb{R}, \mathbb{R})$, where $f(x) = \sin(x)$ and $g(x) = \cos(x)$;

(d) $\{f, g\} \subseteq \text{Abb}(\mathbb{R}, \mathbb{R})$, where $f(x) = e^{rx}$ and $g(x) = e^{sx}$, for fixed $s, r \in \mathbb{R}$.

Solution:

(a) As for exercise 1., we use Gauss elimination on the matrix

$$A := \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & t & t^2 \end{pmatrix}$$

to find the solutions to the linear system

$$A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain that if $t^2 - 2t \neq 0$, i.e. $t \notin \{0, 2\}$, then the matrix transforms to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the vectors are linearly independent in this case.
Now if $t \in \{0, 2\}$, we see that

$$\begin{pmatrix} 2 \\ 4 \\ t^2 \end{pmatrix} = 2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ t \end{pmatrix} \right]$$

(b) The matrix is already in row echelon form. We can use the non-zero value on the diagonal of every column to clear all the columns from right to left. We are then left with a diagonal matrix with non-zero values on the diagonal which we can transform into the identity matrix by dividing every row successively. Then it is clear that the system only has the trivial solution. The whole point of the Gauss algorithm is that it does not change the solution set and therefore we can conclude that the original vectors only have the trivial solution, hence are linearly independent.

Alternative solution: Let $v_1, \ldots, v_n$ be the columns of A, and let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Define $v := \alpha_1 v_1 + \cdots + \alpha_n v_n$. Induction shows that the k-st entry $v^{(k)}$ of v satisfies

$$v^{(k)} = \sum_{i=k}^{n} \alpha_i A_{ki}$$

If $n = 1$, there is nothing to show. So assume that the assertion is true for every upper triangular matrix $B \in M_{n\times n}(\mathbb{R})$. Let $N := n+1 \geqslant 2$ and $A \in M_{N\times N}(\mathbb{R})$ an upper triangular matrix. Then there exists $B \in M_{n\times n}(\mathbb{R})$, a vector $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$A = \begin{pmatrix} B & u \\ 0 & \alpha \end{pmatrix}$$

Let $w_1, \dots, w_n$ be the columns of B and $v_1, \dots, v_N$ the columns of A, and let $v := \sum_{k=1}^{N} \alpha_k v_k$. Take $1 \leqslant k \leqslant n$, then the k-st entry $v^{(k)}$ of v is given by

$$\begin{aligned} v^{(k)} &= \sum_{i=1}^{N} \alpha_i v_i^{(k)} = \sum_{i=1}^{n} \alpha_i w_i^{(k)} + \alpha_N u^{(k)} = \sum_{i=k}^{n} \alpha_i B_{ki} + \alpha_N u^{(k)} \\ &= \sum_{i=k}^{n} \alpha_i A_{ki} + \alpha_N A_{kN} = \sum_{i=k}^{N} \alpha_i A_{ki} \end{aligned}$$

We have $v^{(N)} = \alpha_N A_{NN} = \sum_{k=N}^{N} \alpha_k A_{Nk}$ and so we are done with the induction.

Now let $A \in M_{n\times n}(\mathbb{R})$ be an upper triangular matrix with $A_{ii} \neq 0$ for all $1 \leqslant i \leqslant n$. Let $v_1, \dots, v_n \in \mathbb{R}^n$ be the columns of A and take $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, also $0 = \sum_{i=k}^{n} \alpha_i A_{ik}$ holds for all$1 \leqslant k \leqslant n$. In particular, we have $0 = \alpha_n A_{nn}$, and from $A_{nn} \neq 0$, it follows that $\alpha_n = 0$. Let $1 \leqslant k < n$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$. Our assumption implies $0 = \sum_{i=k}^{n} \alpha_i A_{ki} = \alpha_k A_{kk}$ and thus $\alpha_k = 0$. We get $\alpha_1 = \cdots = \alpha_n = 0$. In other words, the vectors $v_1, \dots, v_n$ are linearly independent.

(c) Assume there exist $a, b \in \mathbb{R}$, one of which is non-zero, such that

$$a\cos(x) + b\sin(x) = 0, \quad \forall x \in \mathbb{R}.$$

Note that if one of a or b vanishes, we directly obtain a contradiction since neither sin nor cos is constant and equal to 0. So, we can assum that $b \neq 0 \neq a$. We then get

$$-\frac{a}{b}\cos(x) = \sin(x), \quad \forall x \in \mathbb{R}.$$

Letting $x = 0$, we obtain $-\frac{a}{b} = 0$, which is once again a contradiction. Hence this set is linearly independent over $\mathbb{R}$.

(d) Assume there exist $a, b \in \mathbb{R}$ and for all $x \in \mathbb{R} : ae^{rx} + be^{sx} = 0$. This is equivalent to $ae^{(r-s)x} = -b$. Hence, either $a = b = 0$ or $e^{(r-s)x}$ is constant. The latter holds if and only if $r = s$. Hence $\{f, g\}$ is linearly independent over $\mathbb{R}$ whenever $r \neq s$, i.e. whenever $f \neq g$.

3. Consider $A_1, A_2 \in M_{2\times 3}(\mathbb{R})$, given by

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Show that $\{A_1, A_2\}$ is linearly independent over $\mathbb{R}$.

(b) Let

$$M := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in M_{2\times 3}(\mathbb{R}) \mid d = e = 0, b - a = f, 3a = c \right\}$$

Prove that $\mathrm{Sp}(A_1, A_2) = M$.

(c) Find $A_3 \in M_{2\times 3}(\mathbb{R})$ such that $\{A_1, A_2, A_3\}$ is linearly independent. Is $\mathrm{Sp}(A_1, A_2, A_3) = M_{2\times 3}(\mathbb{R})$ for any choice of such an A_3?

Solution:

(a) Assume that A_1, A_2 are linearly independent. Then there exist $\alpha, \beta \in \mathbb{R}$, not both 0, such that

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \alpha \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 2\alpha + \beta & 3\alpha \\ 0 & 0 & \alpha + \beta \end{pmatrix} \end{aligned}$$

This computation already shows that we must have $\alpha = \beta = 0$, which is a contradiction. Thus A_1, A_2 are linearly independent.

(b) The calculation in 2.a) implies that $\langle A_1, A_2 \rangle \subseteq M$. It thus remains to show that $M \subseteq \langle A_1, A_2 \rangle$, or in other words, that every matrix $B \in M_{2\times 3}(\mathbb{R})$ of the form

$$\begin{pmatrix} a & b & c \\ 0 & 0 & f \end{pmatrix},$$

where $b - a = f$ and $3a = c$ is contained in $\langle A_1, A_2 \rangle$. We calculate:

$$\begin{aligned} \begin{pmatrix} a & b & c \\ 0 & 0 & f \end{pmatrix} &= \begin{pmatrix} a & a + f & 3a \\ 0 & 0 & f \end{pmatrix} \\ &= \begin{pmatrix} a & 2a & 3a \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & f - a & 0 \\ 0 & 0 & f - a \end{pmatrix} \\ &= aA_1 + (f - a)A_2 \end{aligned}$$

(c) Let us denote

$$A_3 := \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix},$$

We see that for any non-trivial linear combination

$$L = \alpha A_1 + \beta A_2 + \gamma A_3, \text{ with } \alpha, \beta, \gamma \in \mathbb{R},$$

the lower-left entry of L is equal to γc_{21} and therefore will be non-zero when $\gamma \neq 0$ if we choose A_3 so that $c_{21} \neq 0$. We choose

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Similarly, if $\alpha \neq 0$, the upper-right entry of L will not vanish. Hence, for L to be the zero matrix, we must have $\alpha = \gamma - 0$ and $L = \beta A_2 = 0$, which implies $\beta = 0$. So, the set $\{A_1, A_2, A_3\}$ is linearly independent over $\mathbb{R}$. With this choice of A_3, it is clear that $\langle A_1, A_2, A_3 \rangle \neq M_{2\times 3}(\mathbb{R})$ since the middle entry of the second row is 0 for any matrix in this space.
Now assume that $A_3 \in M_{2\times 3}(\mathbb{R})$ is any matrix that such that $\{A_1, A_2, A_3\}$ are linearly independent. Consider a linear combination

$$L = \alpha A_1 + \beta A_2 + \gamma A_3, \text{ with } \alpha, \beta, \gamma \in \mathbb{R}.$$

Keeping the same notation for the entries of A_3 and denoting $L = (l_{ij})_{1\leqslant i\leqslant 2,\, 1\leqslant j\leqslant 3}$, we obtain the following linear system:

$$\begin{cases} \alpha + \gamma c_{11} & = & l_{11} \\ 2\alpha + \beta + \gamma c_{12} & = & l_{12} \\ \vdots & & \\ \gamma c_{21} & = & l_{21} \\ \gamma c_{22} & = & l_{22} \\ \vdots & & \end{cases}$$

So, if $c_{21} = 0$ or $c_{22} = 0$, then clearly $\mathrm{Sp}(A_1, A_2, A_3) \neq M_{2\times 3}(\mathbb{R})$. Otherwise $\gamma = \frac{l_{21}}{c_{21}}$ and therefore $l_{22} = \frac{l_{21}}{c_{21}} c_{22}$. So any matrix that doesn't verify this relation is not in $\mathrm{Sp}(A_1, A_2, A_3)$. So

$$\mathrm{Sp}(A_1, A_2, A - 3) \neq M_{2\times 3}(\mathbb{R}).$$

4. Show that

$$U = \{f \in \mathrm{Abb}(\mathbb{F}_5, \mathbb{F}_5) \mid \sum_{i=0}^{4} f(\bar{i}) = 0\} \subseteq \mathrm{Abb}(\mathbb{F}_5, \mathbb{F}_5)$$

is a linear subspace. Determine a basis of U.

Solution: We write the elements of $\mathbb{F}_5$ as

$$\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}.$$

The function that takes the value 0 on all of $\mathbb{F}_5$ is the zero element in $\mathrm{Abb}(\mathbb{F}_5, \mathbb{F}_5)$ and is contained in U. For $f, g \in U$ and $\mu, \lambda \in \mathbb{F}_5$, we have

$$\sum_{i=0}^{4} (\mu f + \lambda g)(\bar{i}) = \mu \sum_{i=0}^{4} f(\bar{i}) + \lambda \sum_{i=0}^{4} g(\bar{i}) = 0,$$

and thus $\mu f + \lambda g \in U$.

Alternatively, one can directly say that U is the set of sulutions of a homogeneous linear equation and thus a linear subspace.

We will now determine a basis of U. For $i = 1, 2, 3, 4$, consider the map

$$f_i(\bar{j}) = \begin{cases} \overline{4}, & j = 0 \\ \overline{1}, & j = i \\ 0, & \text{else.} \end{cases}$$

Reminder: We have $\overline{4} = -\overline{1}$ in $\mathbb{F}_5$ and thus $f_i \in U$.

These maps are linearly independent. To see this, consider $\lambda_i \in \mathbb{F}_5$ with

$$\sum_{i=1}^{4} \lambda_i f_i = 0.$$

For $j = 1, 2, 3, 4$, this yields

$$0 = \left(\sum_{i=1}^{4} \lambda_i f_i \right) (\bar{j}) = \lambda_j f_j(\bar{j}) = \lambda_j.$$

It remains to show that U is generated by the f_i. To see this, let $f \in U$ and define

$$g := f(\overline{1}) f_1 + f(\overline{2}) f_2 + f(\overline{3}) f_3 + f(\overline{4}) f_4.$$

We get for $i = 1, 2, 3, 4$:

$$g(\bar{i}) = f(\bar{i}) f_i(\bar{i}) = f(\bar{i}).$$

As $f \in U$, we also get

$$f(\overline{0}) = -\left(\sum_{i=1}^{4} f(\bar{i}) \right) = \overline{4} \sum_{i=1}^{4} f(\bar{i}) = g(\overline{0}),$$

and hence $f = g$.

5. Let V be a vector space over some field K that admits a countable basis. Show that every linearly independent subset $S \subseteq V$ is finite or countable.

   Solution: Let $\{v_n \mid n \in \mathbb{Z}_{\geqslant 0}\}$ be a basis of V and let S be a linearly independent subset of V. We denote

$$V_i := \mathrm{Sp}(v_1, v_2, \ldots, v_i) \quad \text{and} \quad S_i := S \cap V_i = \{v \in S \mid v \in V_i\}.$$

   Note that now

$$V = \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad S = S \cap V = \bigcup_{i=1}^{\infty} S_i.$$

Since $\{v_i\}$ is a basis of V, the maximum size of a linearly independent set in V_i is i. Therefore, since all the subsets S_i are linearly independent, we have

$$|S_i| \leqslant i, \; \forall i \in \{1, 2, 3, \dots\}.$$

So, S is a countable union of finite sets, hence countable.

6. Prove that the functions

$$\varphi_a : \mathbb{R}_{>0} \to \mathbb{R}, \quad x \mapsto \frac{1}{a+x}$$

for all $a \in \mathbb{R}_{\geqslant 0}$ are linear independent.

Hint: Use that a non-zero polynomial only has finitely many zeros.

Solution: Consider finitely many $\alpha_i \in \mathbb{R}$ together with pairwise distinct $a_i \in \mathbb{R}_{\geqslant 0}$ such that $\sum_{i=1}^m \alpha_i \varphi_{a_i} = 0$. For all $x \in \mathbb{R}^{>0}$, we get

$$\sum_{i=1}^m \alpha_i \cdot \frac{1}{a_i + x} = 0.$$

Multiplication with $\prod_{i=1}^m (a_i + x)$ yields

$$\sum_{i=1}^m \alpha_i \prod_{j \neq i} (a_j + x) = 0$$

The left side of this equation is a polynomial in x, thus the hint implies that this is the zero polynomial. In particular, the equation also is satisfied for $x = -a_k$ for every $1 \leqslant k \leqslant m$. For all $i \neq k$ we get $\prod_{j \neq i} (a_j - a_k) = 0$; and so the equation reduces to

$$\alpha_k \cdot \prod_{j \neq k} (a_j - a_k) = 0.$$

As the a_i were chosen pairwise distinct, we get $\prod_{j \neq k} (a_j - a_k) \neq 0$ and hence $\alpha_k = 0$. As this is true for all k, we get $\alpha_1 = \cdots = \alpha_m = 0$. In other words, the functions φ_a, $a \in \mathbb{R}_{\geqslant 0}$, are linearly independent.

Multiple Choice questions. Each question can admit several answers.

Question 1. Let V be a vector space over K. Which of the following assertions is true ?

✓ Let $v \in V$, then the set

$$W := \{w \in V \mid \exists \lambda \in K : w = \lambda v\}$$

is a linear subspace of V.
Justification: You can see straight away that this equals

$$\mathrm{Sp}(v) = \{\lambda v \mid \lambda \in K\},$$

by definition, and therefore that it is a linear subspace. Alternatively, you can check it explicitly: we have $0_V = 0 \cdot v$, Moreover, for $w, w' \in W$ with $w = \lambda v$, $w' = \lambda' v$ and for any $\mu, \mu' \in K$, we have

$$\mu w + \mu' w' = \mu\lambda v + \mu'\lambda' v = (\mu\lambda + \mu'\lambda')v.$$

Hence $\mu w + \mu' w' \in W$.

✓ A subset $W \subset V$ is a linear subspace if and only if $\mathrm{Sp}(W) = W$.
Justification: The right-to-left implication follows directly from the fact that the span of a set is linear. On the other hand, if W is linear, we show that $W = \bigcap_{\substack{U \supseteq W \\ U \text{ linear}}} U$. This is indeed the case since otherwise there would exist a linear subspace U such that $W \subseteq U \subsetneq W$, which is a contradiction.

✓ Let $S_1, S_2 \subset V$ be subsets. Then $\mathrm{Sp}(S_1 \cup S_2) = \mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$.
Justification: Left-to-right inclusion: for $i = 1, 2$, we have that $S_i \subseteq \mathrm{Sp}(S_i) \subseteq \mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$. Hence $S_1 \cup S_2 \subseteq \mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$. Since $\mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$ is linear, we obtain $\mathrm{Sp}(S_1 \cup S_2) \subseteq \mathrm{Sp}(S_1) + \mathrm{Sp}(S_2)$.
Right-to-left inclusion: From $S_i \subseteq S_1 \cup S_2$, we have $\mathrm{Sp}(S_i) \subseteq \mathrm{Sp}(S_1 \cup S_2)$ for $i = 1, 2$. Since $\mathrm{Sp}(S_1 \cup S_2)$ is linear, it contains the sum of any 2 of its elements, hence $\mathrm{Sp}(S_1) + \mathrm{Sp}(S_2) \subseteq \mathrm{Sp}(S_1 \cup S_2)$.

✓ Let $S_1, S_2 \subset V$ be subsets. Then $\mathrm{Sp}(S_1 \cap S_2) \subseteq \mathrm{Sp}(S_1) \cap \mathrm{Sp}(S_2)$.
Justification: We have $S_1 \cap S_2 \subseteq S_i \subseteq \mathrm{Sp}(S_i)$, for $i = 1, 2$. Therefore

$$S_1 \cap S_2 \subseteq \mathrm{Sp}(S_1) \cap \mathrm{Sp}(S_2).$$

Since the intersection preserves linearity, the right-hand side above is linear. So, by minimality of the span, we have $\mathrm{Sp}(S_1 \cap S_2) \subseteq \mathrm{Sp}(S_1) \cap \mathrm{Sp}(S_2)$

Question 2. Let V be a vector space and let $S_1, S_2 \subseteq V$ with $S_1 \subsetneq S_2$. Which of the following are true?

(a) If S_1 is a linearly independent set, when is S_2 a linearly independent set?

- ○ Always
- ○ Never
- ✓ Sometimes

Justification: For example, take $V = \mathbb{R}^2$ over $\mathbb{R}$ and let

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Then S_2 is linearly dependent. On the other hand, if we remove the vector $(1,1)$ from S_2, it becomes lineatly independent.

If now S_2 is a linearly independent set, when is S_1 a linearly independent set?

- ✓ Always
- ○ Never
- ○ Sometimes

Justification: Indeed, if there exists a vanishing non-trivial linear combination of vectors of S_1, then the same combination shows that S_2 is linearly dependent since $S_1 \subsetneq S_2$.

(b) Answer the previous question, replacing "linearly independent set" with "generating set for V".

Answer: If S_1 is a generating set for V, then S_2 is **always** a generating set for V. Indeed, we have

$$V \supseteq \mathrm{Sp}(S_2) \supseteq \mathrm{Sp}(S_1) = V.$$

If S_2 is a generating set for V, then S_1 is **sometimes** a generating set for V. For example, take $V = \mathbb{R}^2$ over $\mathbb{R}$ and let

$$S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Then S_1 is still a generating set for V. On the other hand if we remove the second vector from S_1, then it is not a generating set anymore.

(c) Answer question (a), replacing "linearly independent set" with "basis for V".

Answer: If S_1 is a basis of V, then S_2 is **never** a basis for V since it is not linearly independent anymore.

If S_2 is a basis of V, then S_1 is **never** a basis of V since it doesn't generate V anymore.