Musterlösung Serie 7

1. Compute the dimension and find a basis of the following spaces

   (a) The space of upper triangular matrices in $M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$ (for a definition, see Serie 6, exercise 2.b));

   (b) The space of diagonal matrices in $M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$, where diagonal matrices are all matrices $A = (a_{ij})$ such that $a_{ij} = 0$ whenever $i \neq j$;

   (c) The space of symmetric matrices

   $$W = \left\{A \in M_{n\times n}(\mathbb{R}) \mid A^T = A\right\},$$

   where $\cdot^T$ denotes the operation $A = (a_{ij})_{1\leqslant i,j\leqslant n} \mapsto (a_{ji})_{1\leqslant i,j\leqslant n}$;

   (d) The space of matrices $A \in M_{n\times n}(\mathbb{F}_2)$ such that the sum of the columns of A is the null vector.

Lösung:

(a) A matrix is upper triangular if its entries $a_{ij} = 0$ whenever $i > j$. For all $1 \leqslant i, j \leqslant n$ such that $i \leqslant j$, let A_{kl} be the matrix whose entries are zero except for the entry a_{kl}, which is set to 1. Note that the set

$$\{A_{kl} \mid 1 \leqslant k, l \leqslant n \wedge k \leqslant l\}$$

is linearly independent since the matrices do not share any entries and it generates the space of upper triangular matrices. Hence it is a basis. Moreover the cardinality of this set is the sum of natural numbers smaller or equal to n

$$\sum_{m=1}^{n} m = \frac{n(n+1)}{2}.$$

(b) For $1 \leqslant k \leqslant n$, let A_k be the matrix whose entries are all zero, except for the entry a_{kk}, which is set to 1. The set $\{A_k \mid 1 \leqslant k \leqslant n\}$ is linearly independent and generates the space of diagonal matrices. Hence it is a basis. The cardinality of this set is n.

(c) For $1 \leqslant k \leqslant l \leqslant n$, let A_{kl} be the matrix such that $a_{kl} = 1 = a_{lk}$ and all other entries are set to 0. The set $\{A_{kl} \mid 1 \leqslant k \leqslant l \leqslant n\}$ is linearly independent and generates the space of symmetric matrices. We have n matrices with a unique 1 on the diagonal and

$$\sum_{m=1}^{n-1} m = \frac{(n-1)n}{2},$$

additional matrices. Hence the dimension is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

(d) Let W denote the said space. For any $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in W$, we have that for any $1 \leqslant i \leqslant n$,

$$\sum_{k=1}^{n} a_{ik} = 0 \Leftrightarrow a_{in} = \sum_{k=1}^{n-1} a_{ik}.$$

This shows that we have $n-1$ degrees of freedom for each row. Therefore, the dimension of W is $\leqslant n(n-1)$. We attempt to find a basis. For $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n-1$, we define $A_k^{(i)}$ to be the matrix that has a 1 in place of the (i,k)-th and $(i,k+1)$-st entries, and whose other entries vanish. We denote S_i the set $\{A_k^{(i)} \mid 1 \leqslant k \leqslant n-1\}$. To help you visualise it, we write out S_1:

$$S_1 = \left\{ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\}.$$

We first show that any S_i is linearly independent. Consider a vanishing linear combination in S_i. Note that the coefficients in front of $A_1^{(i)}$ and $A_{n-1}^{(i)}$ must vanish since $A_1^{(i)}$ is the only matrix with an entry in the first column and $A_{n-1}^{(i)}$ is the only matrix with an entry in the last column. Repeat the argument in $T \setminus \{A_1^{(i)}, A_{n-1}^{(i)}\}$ for $A_2^{(i)}$ and $A_{n-2}^{(i)}$, and so on. This proves that the only vanishing linear combination in S_i is the trivial one.

Now, let $S = \bigcup_{i=1}^{n} S_i$. We show that S is linearly independent. Assume that we have a vanishing linear combination in S. Since the i-th row of the linear combination vanishes, the combination of elements of S_i vanishes in particular. By the previous paragraph, this implies that the elements of S_i have vanishing coefficients. Since this holds for all $1 \leqslant i \leqslant n$, the only vanishing linear combination in S is the trivial one. Hence S is linearly independent.

Since $\dim(W) \leqslant n(n-1)$ and S is a linearly independent subset of this size, it is a basis.

2. Let W be the linear subspace generated by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Determine a system of equations whose solution is W.

(b) Find all the possible bases of W that can be built using v_1, v_2, v_3, v_4. How many are there?

Solution:

(a) Gaussian Elimination yields that v_1, v_2 and v_4 are linearly independent. The vector $v_3 = v_1 + v_4$ is a linear combination of v_1 and v_4. Hence W is a three-dimensional hyperplane in $\mathbb{R}^4$. It is thus enough to determine the normal vector $n := (n_1, n_2, n_3, n_4) \in \mathbb{R}^4$ which is orthogonal on v_1, v_2, v_4. For this, we solve the system of equations

$$\begin{cases} \langle n, v_1 \rangle &= 0 \\ \langle n, v_2 \rangle &= 0 \\ \langle n, v_4 \rangle &= 0 \end{cases}$$

where $\langle n, v_i \rangle$ denotes the standard scalar product in $\mathbb{R}^4$. This system of equations is equivalent to

$$\begin{cases} n_1 + 2n_2 + n_3 + 2n_4 &= 0 \\ -n_3 + n_4 &= 0 \\ 2n_2 + n_3 + n_4 &= 0 \end{cases}$$

Solving yields that the solutions are given by

$$n = (n_1, n_1, -n_1, -n_1)$$

for an arbitrary $n_1 \in \mathbb{R}$. As the length of a normalvector is irrelevant, we choose $n_1 = 1$ and get $n = (1, 1, -1 - 1)$. Now a system of equations whose set of solutions is W is given by

$$\langle n, x \rangle = 0,$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. This is because all vectors

$$x = (x_1, x_2, x_3, x_4)$$

which satisfy this condition (and thus are solutions of this sytem of equations), are orthogonal to n and so are contained in W. Explicitly, we get the equation

$$x_1 + x_2 - x_3 - x_4 = 0.$$

(b) Because of the relation $v_3 = v_1 + v_4$, the vectors v_1, v_3 and v_4 are linearly dependent, but every two out of the three are linearly independent. The vector v_2 is linearly independent from the three others. Thus v_2 must be contained in any basis which is only allowed to contain the vectors v_1, v_2, v_3, v_4. Thus there are three possibilities:

$$\mathcal{B}_1 = \{v_1, v_2, v_3\}, \quad \mathcal{B}_2 = \{v_1, v_2, v_4\}, \quad \mathcal{B}_3 = \{v_2, v_3, v_4\}.$$

3. Determine a basis and the dimension of the following spaces:

   (a) The set of solutions $S \subseteq \mathbb{R}^3$ of

   $$\begin{aligned} x + y - z &= 0 \\ 3x + y + 2z &= 0 \\ 2x + 3z &= 0 \end{aligned}$$

   (b) $\{0\}$;

   (c) $\{(z, w) \in \mathbb{C}^2 \mid z + iw = 0\}$ as a vector space over $\mathbb{C}$;

   (d) $\{(z, w) \in \mathbb{C}^2 \mid z + iw = 0\}$ as a vector space over $\mathbb{R}$.

   Lösung:

   (a) Gaussian elimination yields as the set of solutions the one-dimensional vector space $V := \left\{\left(-\frac{3}{2}z, \frac{5}{2}z, z\right) \mid z \in \mathbb{R}\right\}$. A basis is for exampled formed by the vector $(-3, 5, 2)$.

   (b) We can write the zero-vector as empty sum, that is, the sum over no elements. Thus the empty set $\varnothing$ forms a basis of $\{0\}$ and the bector space has dimension 0. (Note: The zero-vector 0 can not be contained in any basis, as it is not linearly independent.)

   (c) Let V be the vector space $\{(x, y) \in \mathbb{C}^2 \mid x + iy = 0\}$ over $\mathbb{C}$. Let $(x, y) \in V$. Then we have $x + iy = 0 \Rightarrow y = ix$ and thus $(x, y) = (x, ix) = x(1, i)$ for every $(x, y) \in V$. Hence, we can write each $(x, y) \in V$ as multiple of the vector $(1, i)$. This vector is of course linearly independent in $\mathbb{C}^2$ and thus forms a basis of V. The complex dimension of this vectorspace therefore is 1.

   (d) We have shown in d) that every element in V is of theh form $x(1, i), x \in \mathbb{C}$. When we now consider $x = a + ib, a, b \in \mathbb{R}$, we get

   $$x(1, i) = (a + ib)(1, i) = a(1, i) + ib(1, i) = a(1, i) + b(i, -1).$$

   Hence every $(x, y) \in V$ can be written as linear combination of $(1, i)$ and $(i, -1)$. Those two are linearly independent in V over $\mathbb{R}$ and thus form a basis. Over $\mathbb{R}$, the vector space has dimension 2.

4. Let K be a field, fix $g(X) := X + 5 \in K[X]$, and let $d \geqslant 1$. Compute the dimension of the space

   $$W = \{h \in K[X] \mid \deg(h) \leqslant d \wedge \exists f \in K[X] : h = gf\}.$$

   Lösung: Let

   $$S = \{g, Xg, X^2g, \ldots, X^{d-1}g\}.$$

These polynomials clearly belong to W since they are divisible by g and since

$$\deg(X^i g) = i + \deg(g) = i + 1,$$

which is smaller or equal to d for $0 \leqslant i \leqslant d-1$. Moreover, this is a linearly independent set as all these polynomials are of different degrees. Now, since

$$W \subset K[X]_{\leqslant d} := \{p \in K[X] \mid \deg(p) \leqslant d\},$$

we know that $\dim(W) \leqslant d+1$ (recall that $\{1, X, X^2, \ldots, X^d\}$ is a basis of $K[X]_{\leqslant d}$ over K). Additionally, since there is no constant non-zero polynomial in W, its dimension is at most d. Since S is a set of d linearly independent polynomials in W, it is a basis of W and $\dim(W) = d$.

5. Consider the following subspace of K^n :

$$U := \left\{ (\alpha_1, \ldots, \alpha_n) \in K^n \mid \sum_{i=1}^{n} \alpha_i = 0 \right\},$$
$$D := \{(\alpha, \ldots, \alpha) \in K^n \mid \alpha \in K\}.$$

Determine a basis and compute the dimension of $U, D, U \cap D$, and $U + D$.

Remark: Do not forget to consider the case where K is a field such that $n \cdot 1 = 0$.

Lösung: For $n = 0$ we have $U = D = U \cap D = U + D = 0$. These linear subspaces both have dimension 0.

Now suppose $n \geqslant 1$. Every element of the form $(\alpha, \ldots, \alpha) \in D$ is a multiple of $v := (1, \ldots, 1)$. Since $v \neq 0$, the set $\{v\}$ is a minimal generating set of D or in other words a basis of D. Hence we have $\dim(D) = 1$. Thereafter the $n-1$ vectors

$$v_1 = (1, -1, 0, \ldots, 0), \quad v_2 = (0, 1, -1, 0, \ldots, 0), \quad \ldots \quad v_{n-1} = (0, \ldots, 0, 1, -1)$$

are contained in U and linearly independent. In particular, we have $\dim(U) \geqslant n-1$. As $(1, 0, \ldots, 0) \notin U$, we get $\dim(U) < \dim(K^n) = n$ and thus find that $\dim(U) = n-1$. As the vectors $v_1, \ldots, v_{n-1}$ are linearly indepnendent and their number is equal to the dimension of U, they form a basis of U.

If we have $n \cdot 1 \neq 0$ in K, then $v = (1, \ldots, 1) \notin U$. In that case, we have $U \cap D = 0$, and hence $\varnothing$ $\dim(U \cap D) = 0$. As $U = \langle v_1, \ldots, v_{n-1} \rangle$ we also find that $v, v_1, \ldots, v_{n-1}$ are linearly independent. This shows $\dim(U + D) \geqslant n$. But we already have $\dim(U + V) \leqslant \dim(K^n) = n$, and consequently conclude that $U + D$ has dimension n and admits the basis $\{v, v_1, \ldots, v_{n-1}\}$. Of course the standard basis of K^n is also a basis of $U + D$

If we have $n \cdot 1 = 0$ in K, then $v = (1, \ldots, 1) \in U$, and hence $D \subset U$. In that case we have $U \cap D = D$ and $U + D = U$.

6. Determine a basis and compute the dimension of the space

$$\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \wedge (f + f'' = 0)\}.$$

Hint: Note that f, f', and f'' exist and are continuous. Moreover, you can use the following result from Analysis: the function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} : f(x) = 0$ is the unique continuous solution to

$$\begin{cases} f + f'' &= \quad 0 \\ f(0) &= \quad f'(0) \quad = \quad 0 \end{cases}$$

Solution: To put the hint into different words : for every two times continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0, f'(0) = 0$ and $f(x) + f''(x) = 0 \forall x \in \mathbb{R}$, we have $f(x) = 0 \forall x \in \mathbb{R}$. Now let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } C^2 \text{ and } f + f'' = 0\}$ and let $f \in V$. Set $a := f(0)$ und $b := f'(0)$. Also define $g(x) := f(x) - a\cos(x) - b\sin(x)$. We have $g(0) = 0, g'(0) = 0$ and $g \in V$ as $\sin(x), \cos(x) \in V$. Thus, we have $g(x) = 0 \forall x \in \mathbb{R}$. Therefore, we find $f(x) = a\cos(x) + b\sin(x)$. Thus every element of V is a linear combination of $\sin(x)$ and $\cos(x)$. These functions are linearly independent elements over $\mathbb{R}$ and thus form a basis of V. We conclude that V has dimension 2.