Musterlösung Serie 8

1. Let F be a field. Find a linear complement of the following subspaces in $M_{n\times n}(F)$ (see Serie 7 for definitions):

   (a) The subspace of upper triangular matrices;

   (b) The subspace of symmetric matrices.

   Solution:

   (a) Let us denote U the subspace of upper triangular matrices and denote W the subspace of strictly lower triangular matrices, i.e W is the set of all matrices $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ such that $a_{ij} = 0$ whenever $i \leqslant j$. We let the reader check that this is indeed a linear subspace of $M_{n\times n}(F)$. Note that any matrix can be written as the sum of an upper triangular matrix and a strictly lower triangular matrix. As an illustration

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}.$$

   This shows that $U + W = M_{n\times n}(F)$. Moreover, since matrices from U and matrices from W do not have any non-zero entry in common, we do also have $U \cap W = \{0\}$. This shows that W is a complement of U in $M_{n\times n}(F)$.

   (b) Denote U the subspace of symmetric matrices and consider once again the space of strictly lower triangular matrices, denoted W. We show that $U+W = M_{n\times n}(F)$. Let $C = (c_{ij})_{1\leqslant i,j\leqslant n} \in M_{n\times n}(F)$. Define $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ to be the symmetric matrix such that

$$a_{ij} = c_{ij}, \quad \text{when } i \leqslant j.$$

   Also define $B = (b_{ij})_{1\leqslant i,j\leqslant n}$ to be the strictly lower triangular matrix such that

$$b_{ij} = c_{ij} - c_{ji}, \quad \text{when } i > j.$$

   Now, we do have $C = A + B$.

   Each symmetric matrix that is not the 0 matrix admits a non-vanishing entry on the diagonal or over the diagonal, while this is not the case for any matrix of W. This shows that $U \cap W = \{0\}$.

2. Let $b, c \in \mathbb{R}$ and let $\mathbb{R}[x]$ denote the space of polynomial functions of 1 variable with coefficients in $\mathbb{R}$. Define $T : \mathbb{R}[x] \to \mathbb{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + cp(0)^2\right).$$

Show that T is linear if and only if $b = c = 0$.

Solution: If $b = c = 0$, we have

$$Tp = \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)dx\right).$$

Note that evaluation is also linear: let $p, q \in \mathbb{R}[x]$ and let $\mu, \nu, a \in \mathbb{R}$. Then

$$(\mu p + \nu q)(a) = \mu p(a) + \nu q(a).$$

Derivation is also a linear map $\mathbb{R}[x] \to \mathbb{R}[x]$:

$$(\mu p + \nu q)'(x) = (\mu p)'(x) + (\nu q)'(x) = \mu p'(x) + \nu q'(x).$$

Finally,

$$\int_{-1}^{2} x^3(\mu p + \nu q)(x)dx = \mu \int_{-1}^{2} x^3 p(x)dx + \nu \int_{-1}^{2} x^3 q(x)dx.$$

Therefore T is linear.

Assume now that $b \neq 0$. Let $p, q \in \mathbb{R}[x]$ and $\mu, \nu \in \mathbb{R}$. Then

$$\begin{aligned}(p+q)(1)(p+q)(2) &= (p(1) + q(1))(p(2) + q(2)) \\ &= p(1)p(2) + p(1)q(2) + q(1)p(2) + q(1)q(2).\end{aligned}$$

The right-hand side is in general different from $p(1)p(2) + q(1)q(2)$. Therefore the first factor is not linear when $b \neq 0$.

If finally $c \neq 0$, note that since

$$(p+q)(0)^2 = (p(0) + q(0))^2 = p(0)^2 + 2p(0)q(0) + q(0)^2$$

the second factor is not linear. This shows the equivalence.

3. Suppose that U and V are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exists 2 vectors in $U \cap V$ such that neither of these vectors is a scalar multiple of the other.

Solution: Assume that $U \cap V$ is one dimensional, generated by w. We can first extend $\{w\}$ to a basis of U. Let us note it $\mathcal{B}_1 = \{w, u_1, u_2, u_3\}$. In parallel, extend $\{w\}$ to a basis of V denoted $\mathcal{B}_2 = \{w, v_1, v_2, v_3\}$. Finally, the set

$$\mathcal{B}_1 \cup \mathcal{B}_2 = \{w, u_1, u_2, u_3, v_1, v_2, v_3\}$$

is a basis of $U \cup V$. However, the above set contains 7 distinct vectors, so since $U + V \subseteq \mathbb{C}^6$, which is 6-dimensional, we obtain a contradiction.

Aliter: We can apply the formula

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

We have $U + V \subseteq \mathbb{C}^6$, hence

$$6 \geqslant \dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V) = 8 - \dim(U \cap V).$$

We conclude that $\dim(U \cap V) \geqslant 2$ and therefore that there must exist 2 linearly independent vectors in $U \cap V$.

4. Suppose that $\{v_1, \ldots, v_m\}$ is linearly independent in V and let $w \in V$. Prove that

$$\dim \operatorname{Sp}(v_1 + w, v_2 + w, \ldots, v_m + w) \geqslant m - 1.$$

Solution: Consider the set

$$\begin{aligned} &\{v_1 + w, v_2 + w - (v_1 + w), \ldots, v_m + w - (v_1 - w)\} \\ =&\{v_1 + w, v_2 - v_1, \ldots, v_m - v_1\}. \end{aligned}$$

We show that $\{v_2 - v_1, \ldots, v_m - v_1\}$ is linearly independent. Indeed, assume that there exists $\{a_i \mid 2 \leqslant i \leqslant m\}$ such that

$$\begin{aligned} &\sum_{i=2}^{m} a_i(v_i - v_1) = 0 \\ \iff \quad & -\left(\sum_{i=2}^{m} a_i\right) v_1 + \sum_{i=2}^{m} a_i v_i = 0. \end{aligned}$$

Since the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent, we must have

$$a_i = 0, \quad 2 \leqslant i \leqslant m.$$

So, we have proved that there is a set of $m - 1$ linearly independent vectors in $\operatorname{Sp}(v_1 + w, v_2 + w, \ldots, v_m + w)$. Thus, we obtain the result.

5. Let V be a vector space over a field F and consider 3 linear subspaces U_1, U_2, U_3 such that $V = U_1 + U_2 + U_3$ and for $i, j \in \{1, 2, 3\}$ such that $i \neq j$, we have $U_i \cap U_j = \{0\}$.

   Is it true that for any $v \in U_1 + U_2 + U_3$ there exists a unique triple (u_1, u_2, u_3) with $u_i \in U_i$ such that $v = u_1 + u_2 + u_3$?

Solution: This is not true in general, we give a counterexample. Consider a field F and the spaces

$$\begin{aligned} U_1 &= \{(x, y, 0) \in F^3 \mid x, y \in F\}, \\ U_2 &= \{(0, 0, z) \in F^3 \mid z \in F\}, \\ U_3 &= \{(0, y, y) \in F^3 \mid y \in F\}. \end{aligned}$$

Clearly $F^3 = U_1 + U_2 + U_3$ since for any $x, y, z \in F$

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0).$$

We do have $U_i \cap U_j = \{0\}$ whenever $i \neq j$. However,

$$\begin{aligned} (0, 0, 0) &= (0, 0, 0) + (0, 0, 0) + (0, 0, 0) \\ (0, 0, 0) &= (0, 1, 0) + (0, 0, 1) + (0, -1, -1). \end{aligned}$$

6. Let V be a finite-dimensional vector space over a field F and let

$$V \supseteq U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots$$

be an infinite sequence of nested subspaces.

(a) Show that this sequence stabilises, i.e. show that there exists a $N \in \mathbb{N}$ such that for all $n \geqslant N : U_n = U_N$.

(b) Is this still the case if we now assume that V is infinite-dimensional?

(c) Assume that V is infinite-dimensional and that $\dim U_n \geqslant 1$ for all $n \in \mathbb{N}$. What can you say about $\bigcap_{n\in\mathbb{N}} U_n$?

Solution:

(a) Let $d := \dim(V)$, since the subspaces in the given sequence are nested, we obtain the following decreasing sequence in $\mathbb{N}$:

$$d \geqslant \dim(U_0) \geqslant \dim(U_1) \geqslant \dim(U_2) \geqslant \cdots \geqslant \dim(U_k) \geqslant \cdots$$

This is a monotonically decreasing sequence of natural numbers, hence it is bounded form below by 0 and has to converge. Namely, there exists $e \in \mathbb{N}$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$: $e = \dim(U_N) = \dim(U_n)$. We conclude that for any such n we have $U_N = U_n$.

(b) Not necessarily, the example developed in c) gives a counter example.

(c) For any vector space V, the constant sequence $U_n = V$ has intersection V.

Even though the dimension is bounded from the bottom, the intersection might be $\{0\}$. Indeed, consider the space $F[X]$ of polynomials with coefficients in F. Let $U_0 = F[X]$ and let

$$U_i = \mathrm{Sp}(\{X^p \mid p \geqslant i\})$$

For any $i \geqslant 0$, we do have $\dim U_i \geqslant 1$ since $X^i \in U_i$. Now, let $N \geqslant 0$. Then

$$\bigcap_{n=0}^{N} U_n = U_N$$

since the subspaces are nested. However, for any polynomial $p \in F[X] \setminus \{0\}$, there exists $d \geqslant 0$ such that $p \notin U_d$ (this will hold for any $d > \deg(p)$). This implies

$$\bigcap_{n \geqslant 0} U_n = \{0\}.$$

Multiple Choice Fragen. More than one answer can be correct.

Frage 1. Which of the following maps are linar?

- ✓ $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (x + y, 2x, 0)$
- $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (x + y, 2x, 0)$
- ✓ $f : K^3 \to K^2$, $(x, y, z) \mapsto (\alpha x + \beta y + \gamma z, \delta x + \varepsilon y + \eta z)$ for fixed $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ in the field K

Frage 2. Which of the linear maps below can be written as $x \mapsto Ax$, where A is the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$$

- ✓ $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (2x + y, x, 2y)$
- $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (2x + y, x + 2z)$