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Musterlösung Serie 9

1. Show that if V is a one-dimensional vector space over a field K and $T \in \mathrm{Hom}_K(V, V)$, then there exists $\lambda \in K$ such that for all $v \in V : Tv = \lambda v$. Explain then why an isomorphism $V \to K$ depends on a choice of basis, while one from $\mathrm{Hom}_K(V, V)$ to K doesn't.

   Solution: Let $v_0 \in V \setminus \{0\}$. Since V is one-dimensional, $\{v_0\}$ is a basis of V. We have $Tv_0 \in V$, so there exists $\lambda \in K$ such that $Tv_0 = \lambda v_0$. Now let $v_1 \in V$. For the same reason, there exists $\mu \in K$ such that $v_1 = \mu v_0$. By linearity of T, we then have
$$Tv_1 = T(\mu v_0) = \mu T(v_0) = \mu\lambda v_0 = \lambda(\mu v_0) = \lambda v_1.$$
   Since v_1 was arbitrary, this proves the statement.

   Aliter: You have seen in the lectures that such a linear map T can be represented by a matrix. Since a 1×1 matrix is a scalar, this proves the statement.

   Regarding the second part of the exercise, note that if we want to define a linear isomorphism from V to K, we need to first specify the image of a certain basis. Such a homomorphism is called "non-canonical". On the other hand, we can define a linear isomorphism
$$\begin{array}{ccl} \mathrm{Hom}(V, V) & \to & K \\ T & \mapsto & \lambda, \text{ where } \lambda \text{ is such that } \forall v \in V : Tv = \lambda v. \end{array}$$
   In this case, we did not need to pick a basis in order to define the isomorphism. Such a homomorphism is called "canonical".

2. Denote $\mathbb{R}[x]_d$ the set of polynomials over $\mathbb{R}$ of degree lower or equal to d. Suppose that $D \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[x]_3, \mathbb{R}[x]_2)$ is the differentiation map $Dp = p'$. Find a basis of $\mathbb{R}[x]_3$ and a basis of $\mathbb{R}[x]_2$ such that the matrix of D with respect to these bases is
$$\begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
   Solution: Consider the basis $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ of $\mathbb{R}[x]_3$ and the basis $\{1, 1+x, 1+x+x^2\}$ of $\mathbb{R}[x]_2$. We have
$$\begin{array}{ccccc} D(1) & = & 0 & = & 0 \cdot 1 + 0 \cdot (1+x) + 0 \cdot (1+x+x^2) \\ D(1+x) & = & 1 & = & 1 \cdot 1 + 0 \cdot (1+x) + 0 \cdot (1+x+x^2) \\ D(1+x+x^2) & = & 1+2x & = & -1 \cdot 1 + 2 \cdot (1+x) + 0 \cdot (1+x+x^2) \\ D(1+x+x^2+x^3) & = & 1+2x+3x^2 & = & -1 \cdot 1 + -1 \cdot (1+x) + 3 \cdot (1+x+x^2) \end{array}$$
   Hence the matrix of D is the desired one with respect to these two bases.

3. Let V, W be vector spaces over a field K. Suppose that $U \subsetneq V$ is a linear subspace and let S be a non-trivial element of $\mathrm{Hom}_K(U, W)$ (i.e. we assume that S does not map everything to 0). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv, & \text{if } v \in U \\ 0, & \text{if } v \in V \setminus U \end{cases}$$

Is T a linear map?

Solution: Let $u \in U$ such that $Su \neq 0$. Such a u exists by our assumption on S. Let also $v \in V \setminus U$ (since $U \subsetneq V$, this set is non-empty). We then observe that $u + v \notin U$. Indeed, if it were the case, then there would exist some $u' \in U$ such that $u + v = u' \Leftrightarrow v = u' - u \in U$, which is a contradiction to our choice of v. Hence,

$$T(u + v) = 0 \neq Su + 0 = Tu + Tv,$$

which shows that T isn't linear.

4. Let U, V, W be vector spaces over a field K and let $T : V \to W$ and $S : W \to U$ be linear maps.

   (a) Prove that
   $$\mathrm{rank}(S \circ T) \leqslant \min(\mathrm{rank}(S), \mathrm{rank}(T)).$$

   (b) Show that $\mathrm{rank}(S \circ T) = \mathrm{rank}(S)$ whenever T is surjective.

   (c) Show that $\mathrm{rank}(S \circ T) = \mathrm{rank}(T)$ whenever S is injective.

Solution:

(a) Note that since T is linear, $\mathrm{Bild}(T)$ is a subspace of W. We define the restriction of S to $\mathrm{Bild}(T)$ to be the linear map

$$S|_{\mathrm{Bild}(T)} : \begin{array}{ccc} \mathrm{Bild}(T) & \to & U \\ v & \mapsto & S(v) \end{array}$$

Now note that $\mathrm{Bild}(S \circ T) = \mathrm{Bild}(S|_{\mathrm{Bild}(T)}) \subset \mathrm{Bild}(S)$. Hence

$$\mathrm{rank}(S \circ T) = \mathrm{rank}(S|_{\mathrm{Bild}(T)}) \leqslant \mathrm{rank}(S). \tag{1}$$

On the other hand,

$$\mathrm{rank}(T) = \dim(\mathrm{Bild}(T)) = \dim(\mathrm{Kern}(S|_{\mathrm{Bild}(T)})) + \mathrm{rank}(S|_{\mathrm{Bild}(T)}). \tag{2}$$

Therefore,

$$\mathrm{rank}(S \circ T) = \mathrm{rank}(S|_{\mathrm{Bild}(T)}) \leqslant \mathrm{rank}(T).$$

This shows

$$\mathrm{rank}(S \circ T) \leqslant \min\{\mathrm{rank}(S), \mathrm{rank}(T)\}.$$

(b) If T is surjective, the subspace $\text{Bild}(T)$ is now the whole of W. Then

$$S|_{\text{Bild}(T)} = S|_W = S$$

and by (1), we have

$$\text{rank}(S \circ T) = \text{rank}(S|_{\text{Bild}(T)}) = \text{rank}(S).$$

(c) If S is injective, we have $\dim(\text{Kern}(S)) = 0$, so $\dim(\text{Kern}(S|_{\text{Bild}(T)})) = 0$. Hence, by (2), we obtain

$$\text{rank}(T) = \text{rank}(S|_{\text{Bild}(T)}) = \text{rank}(S \circ T).$$

5. Let V be a vector space. An Endomorphism $P : V \to V$ satisfying $P^2 := P \circ P = P$ is called idempotent or a projection. Show:

   (a) For ever projection P, its image $\text{Bild}(P)$ is a linear complement of $\text{Kern}(P)$ in V.

   (b) For any subvectorspaces $W_1, W_2 \subset V$, such that W_1 is a complement of W_2 in V, there exists a unique projection $P : V \to V$ with

$$\text{Kern}(P) = W_1 \quad \text{und} \quad \text{Bild}(P) = W_2.$$

Lösung:

(a) Let $P\colon V \to V$ be a projection and define

$$W_1 := \text{Kern}(P) \quad \text{and} \quad W_2 := \text{Bild}(P)\,.$$

We need to show that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$.
For any $v \in V$ we have

$$P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0,$$

hence $v - P(v) \in \text{Kern}(P) = W_1$. Moreover, we have $P(v) \in \text{Bild}(P) = W_2$. Thus, we get

$$v = (v - P(v)) + P(v) \in W_1 + W_2,$$

and $W_1 + W_2 = V$.
Now let $v \in W_1 \cap W_2$. Since v lies in the image of P, there exists a $w \in V$ with $P(w) = v$. Application of P on both sides of the equation yields $v = P(w) = P^2(w) = P(v)$. As v is also contained in the kernel of P, we have $v = P(v) = 0$. Thus, we get $W_1 \cap W_2 = \{0\}$.

(b) Let $W_1, W_2 \subseteq V$ be any subvectorspaces of V satisfying $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. For any $v \in V$ there exist unique $w_1 \in W_1$ and $w_2 \in W_2$ such that we have $v = w_1 + w_2$. Consider the map $P\colon V \to V$, which maps $v \in V$ to this $w_2 \in W_2$. One can show by explicit computations that P is linear and a projection with $W_1 = \mathrm{Kern}(P)$ and $W_2 = \mathrm{Bild}(P)$. We leave this to the reader.

It remains to show that P is unique. Let P' be another projection with $W_1 = \mathrm{Kern}(P')$ and $W_2 = \mathrm{Bild}(P')$. Then we have

$$P|_{W_1} = 0 = P'|_{W_1}.$$

For an arbitrary element $v \in W_2$ there exist $w, w' \in V$ such that $P(w) = v$ and $P'(w') = v$. This yields

$$P(v) - P'(v) = P(P(w)) - P'(P'(w')) = P(w) - P'(w') = v - v = 0$$

and hence we get $P|_{W_2} = P'|_{W_2}$. As we have $V = W_1 + W_2$ it follows that $P = P'$.

6. Let $f : V \to W$ be a linear map of K-vector spaces. Show:

(a) For every subvectorspace $W' \subset W$ the preimage

$$f^{-1}(W') := \{v \in V \mid f(v) \in W'\}$$

is a subvectorspace of V.

(b) We have

$$\dim f^{-1}(W') = \dim \mathrm{Kern}(f) + \dim(\mathrm{Bild}(f) \cap W').$$

Solution: Let $V' := f^{-1}(W')$.

(a) We need to show that V' is non empty and that for arbitrary $x, y \in V'$ and $\alpha \in K$ the sum $x + y$ and the product αx is again contained in V'.

As $f(0) = 0 \in W'$, the vector 0 is contained V' and thus V' is non empty. Linearity of f yields

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(\alpha x) = \alpha f(x).$$

As $f(x)$, $f(y) \in W'$, we get folgt from the axioms of subvectorspaces that $f(x) + f(y)$ and $\alpha f(x)$ are again contained in W'. Therefore, we have that $x + y$ and αx are contained in V'.

Note. Is it also the case for the image of a linear subspace via a linear map?

(b) The definition of V' yields that there exists a map

$$f'\colon V' \to W',\ v' \mapsto f'(v') := f(v).$$

As f is linear, the newly defined f' is linear as well. We get

$$\dim(V') = \dim(\operatorname{Kern}(f')) + \dim(\operatorname{Bild}(f')). \tag{3}$$

Since $0 \in W'$, it follows that $\operatorname{Kern}(f) \subset V'$. Plugging in the definition, we get $\operatorname{Kern}(f') = \operatorname{Kern}(f)$. Moreover, we have

$$\begin{aligned}\operatorname{Bild}(f') &= \{w \in W' \mid \exists v \in V' : f'(v) = w\} \\ &= \{w \in W' \mid \exists v \in V : f(v) = w\} \\ &= \{w \in W \mid \exists v \in V : f(v) = w \text{ und } w \in W'\} \\ &= \operatorname{Bild}(f) \cap W'.\end{aligned}$$

Plugging this into (3), we get

$$\dim(V') = \dim(\operatorname{Kern}(f)) + \dim(\operatorname{Bild}(f) \cap W').$$

Exercises that will not be presented

7. Let V, W be vector spaces over a field K and let $T : V \to W$ be an isomorphism of vector spaces. Show that:

   (a) T maps linearly independent sets to linearly independent sets;

   (b) T maps spanning sets of V to spanning sets of W;

   (c) T maps bases to bases.

   Lösung:

   (a) Assume that $\{v_i \mid 1 \leqslant i \leqslant n\}$ is linearly independent. We prove that $\{T(v_i)\}$ is linearly independent. Assume that there exists $\{a_i \mid 1 \leqslant i \leqslant n\} \subset K$ such that

   $$\sum_{1\leqslant i\leqslant n} a_i T(v_i) = 0 \Leftrightarrow T\left(\sum_{1\leqslant i\leqslant n} a_i v_i\right) = 0.$$

   This is equivalent to $\sum_{1\leqslant i\leqslant n} a_i v_i = 0$ since T is an isomorphism. Since we assumed $\{v_i \mid 1 \leqslant i \leqslant n\}$ to be linearly independent, this is equivalent to $\forall i : a_i = 0$. Summing up, we have

   $$\sum_{1\leqslant i\leqslant n} a_i T(v_i) = 0 \Leftrightarrow \forall 1 \leqslant i \leqslant n : a_i = 0.$$

   Hence, $\{T(v_i) \mid 1 \leqslant i \leqslant n\}$ is linearly independent.

(b) Assume that $S = \{v_1, v_2, \ldots, v_r\}$ is a spanning set of V. Let w be an arbitrary element of W. Since T is an isomorphism, w admits a unique preimage v in V. Since $V = \mathrm{Sp}(v_1, \ldots, v_r)$, we can write v uniquely as $v = \sum_{1 \leqslant i \leqslant r} a_i v_i$ for $\{a_i \mid 1 \leqslant i \leqslant r\} \subset K$. Therefore,

$$w = T(v) = T\left(\sum_{1 \leqslant i \leqslant r} a_i v_i\right) = \sum_{1 \leqslant i \leqslant r} a_i T(v_i)$$

Since w was arbitrary, this shows $W = \mathrm{Sp}(\{T(v_i) \mid 1 \leqslant i \leqslant r\})$.

(c) Since a basis is a linearly independent spanning subset. The combination of (a) and (b) shows (c).

8. Let V, W be vector spaces over $\mathbb{Q}$. We say that a map $f : V \to W$ is additive, if

$$\forall x \in V \, \forall y \in V : f(x+y) = f(x) + f(y).$$

Show that

$$\mathrm{Hom}_{\mathbb{Q}}(V, W) = \{f : V \to W \mid f \text{ is additive}\}.$$

Solutions: We want to show that any additive map $f : V \to W$ also satisfies $\forall q \in \mathbb{Q}, \forall v \in V : f(qv) = qf(v)$. Let $v \in V$. We first note that

$$f(0) = f(0+0) = f(0) + f(0) \implies f(0) = 0.$$

It directly follows that $0 = f(v + (-v)) = f(v) + f(-v)$. Hence, $f(-v)$ is the additive inverse of $f(v)$ in W.

We continue by showing by induction that for any $m \in \mathbb{N}$, $f(mv) = mf(v)$. The above shows the base step $m = 0$. We now assumes that it is proved for all m with $0 \leqslant m \leqslant k$ and we prove it for $k+1$. We have

$$f((k+1)v) = f(kv + v) = f(kv) + f(v) = kf(v) + f(v) = (k+1)f(v),$$

where we used the induction hypothesis to obtain the second-to-last equality. Now, for any negative integer $n \in \mathbb{Z}$, we do have $-n \in \mathbb{N}$. So,

$$f(nv) = f((-n)(-v)) = (-n)f(-v) = (-n)(-f(v)) = nf(v).$$

We have shown that $f(nv) = nf(v)$ for all $n \in \mathbb{Z}$.

Finally, notice that for any $n \in \mathbb{Z} \setminus \{0\}$,

$$\frac{n}{n} f(v) = f(v) = f\left(\frac{n}{n} v\right) = nf\left(\frac{1}{n} v\right) \Leftrightarrow n\left(\frac{1}{n} f(v) - f\left(\frac{1}{n} v\right)\right) = 0.$$

This is equivalent to $\frac{1}{n} f(v) = f\left(\frac{1}{n} v\right)$.

We conclude that for all $m \in \mathbb{Z}$, for all $n \in \mathbb{Z} \setminus \{0\}$, for all $v \in V$:

$$f\left(\frac{m}{n}v\right) = mf\left(\frac{1}{n}v\right) = \frac{m}{n}f(v).$$

Since any $q \in \mathbb{Q}$ can be written as $q = \frac{m}{n}$ for some $m \in \mathbb{Z}$ and some $n \in \mathbb{Z} \setminus \{0\}$, we have proved that f is $\mathbb{Q}$-linear.

Multiple Choice Questions. More than one answer can be correct.

Question 1. Let $\mathcal{A}$ and $\mathcal{B}$ be bases of $\mathbb{R}^2$ and denote $\{e_1, e_2\}$ the standard basis. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to $\mathcal{A}$ as a basis of the domain and $\mathcal{B}$ as a basis of the codomain. Which of the following statements are true?

- ✓ If $\mathcal{A} = \mathcal{B} = \{e_1, e_2\}$, f is a rotation around the origin.
- ✓ If $\mathcal{A}$ is the standard basis and $\mathcal{B} = \{e_2, -e_1\}$, f is a symmetry with respect ot the point $(0, 0)$.
- ✓ If $\mathcal{A}$ is the standard basis and f is the identity, then $\mathcal{B} = \{-e_2, e_1\}$.
- ✓ If $\mathcal{B}$ id the standard basis and f is the symmetry with respect to the y-axis, then $A = \{e_2, e_1\}$.

Question 2. Which of the following statements are true?

- ✓ Let V be an n-dimensional vector space over $\mathbb{R}$. The map $V \to \mathbb{R}^n$, $v \mapsto [v]_{\mathcal{B}}$, that sends any vector $v \in V$ to its coordinate vector with respect to a basis $\mathcal{B}$ of $\mathbb{R}^n$ is linear.
- • Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $\mathrm{Kern}(f) \neq \{0\}$ and $\mathrm{Bild}(f) \neq \{0\}$. Then there exists a non-trivial vector v that is in the kernel and in the image of f.
- ✓ If the kernel of a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is trivial, the map is invertible.